CHAPTER 4: Mathematical Proof

> Faith is different from proof; the one is human, the other is a gift of God. *Justus ex fide vivit.* It is this faith that God Himself puts into the heart.[21]
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> Intuitive minds, on the contrary, being thus accustomed to judge at a single glance, are so astonished when they are presented with propositions of which they understand nothing, and the way to which is through definitions and axioms so sterile, and which they are not accustomed to see thus in detail, that they are repelled and disheartened.[22]

This chapter deals with one of the most important aspects of the activities of mathematicians. In this section, we deal with how theorems are proven and how mathematicians decide what statements they should try to prove. Analogy and induction are useful in discovering theorems. But when it comes to proofs, analogy and induction are inadequate. It is the view of modern mathematics that proofs must be deductive.

The father of mathematics approached this way is Thales, who lived about 600 BC. Thales is the author of the first recorded proofs of some theorems in geometry. He used a deductive argument to prove that a complicated statement must be true because it was the logical consequence of some simpler statements which other people accepted as true. During the next few centuries, this approach to doing mathematics was formalized into the axiomatic method.

The Axiomatic Method consists of four components:

1. Definitions
2. Rules of logic
3. Axioms: self–evident truths
4. Theorems: additional truths deduced from the Axioms

These are the four components of an axiomatic system. We will examine them in detail when we discuss Euclidean geometry.

There are both advantages and disadvantages to requiring mathematical statements to be proven deductively. One disadvantage is that the content of mathematics is limited to what can be proved deductively. Text books in other disciplines would be much shorter if all but the deductively proved statements were eliminated.

There is an advantage, however, which goes a long way to offsetting the above disadvantage. Theorems which are proved deductively are as true as the assumptions on which they are based. Thus while in mathematics we may not assert as much as one would hope, we are certain about what we do assert.

Below are some examples of what constitutes a deductive proof, as well as some examples of non-proofs.

Direct Proof

Theorem 1: When you add an integer to itself, you get an even integer.

Not a Proof: 1 + 1 = 2, 2 + 2 = 4, and 3 + 3 = 6, so it must always work.

Proof: Let n be an integer. Then when you add n to itself, you get n + n. This is 2 • n, which is even.

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[21]Pascal, Pensees, p. 91.

[22]Pascal, Pensees,p. 8.

As a more complicated example of a proof, consider the following theorem. By a multiple of a Pythagorean triple, I mean the result of multiplying each member of a Pythagorean triple by the same positive integer. For instance, multiplying 3, 4, 5 by 7 yields the triple 21, 28, 35.

Theorem 2: A multiple of a Pythagorean triple is itself a Pythagorean triple.

Proof: Suppose a, b, c is a Pythagorean triple. Let m be a positive integer, and consider ma, mb, and mc. $(ma)^2 + (mb)^2 = m^2 a^2 + m^2 b^2 = m^2 (a^2 + b^2) = m^2 (c^2) = (mc)^2$. So ma, mb, mc is a Pythagorean triple.

<u>Indirect Proof</u>

Theorem 3: If the product of an integer with itself is even, then the integer itself is even.

Not a proof: $2 \cdot 2 = 4$, $4 \cdot 4 = 16$, whereas $3 \cdot 3 = 9$.

Proof: We are given that the <u>product</u> of an integer with itself is <u>even</u>. Suppose for the moment that the <u>integer</u> <u>itself</u> were <u>odd</u>. Then you would have
$n = 2b + 1$ for some integer b. So
$n \cdot n = n^2 = (2b + 1)^2 = 4b^2 + 4b + 1 = 2(2b^2 + 2b) + 1$, which is odd. This is a contradiction of the original assumption that the product of the integer with itself was even. So the integer itself must be even.

In the above cases, the difference between the proof and the non-proof is the difference between deduction and induction. Induction might suggest that the pattern would always work, but the general argument by deduction guarantees it.

An indirect proof works by using the fact that mathematics cannot contain contradictions. Therefore, if assuming a statement to be true leads to a contradiction, then that statement must in fact be false.

As we said in the previous chapter, this is in essence the general line of reasoning that Paul uses in 1 Corinthians 15 to argue against those who said "that there is no resurrection of the dead" (v. 12). He assumes for the sake of argument that "there is no resurrection of the dead" (v. 13). It follows that since Jesus was dead, he has not been resurrected (v. 13,16). However, "Christ has indeed been raised from the dead" (v. 20; vv. 5-8 give the historical evidence). Hence the original premise (which someone was apparently preaching in Corinth) that there is no resurrection of the dead must be false.

A rather sophisticated mathematical use of this type of reasoning is Euclid's proof that there are an infinite number of prime numbers. What I mean by an "infinite" number of prime numbers will be discussed in greater detail later. For the moment it will suffice to understand this to mean that if someone thought that the list of all prime numbers would be finite, that person would be wrong. The indirect argument is a bit tricky, and the details require careful attention, but both the method and the result are important for our purposes, so we will do it. But first we need a little information about prime numbers.

A number is **divisible** by a second number if there is no remainder when you divide the number by the second number. For instance, 12 is divisible by 3. On the other hand, 12 is not divisible by 5. A number is **prime** if it is divisible by no number except itself and 1. For instance, 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29 are the prime numbers less than 30.

One theorem which can be proven is that if a number is not prime, then it is divisible by a prime number. This implies that if you want to check to see whether a number is prime, all you need to do is to see if it is divisible by a prime number. In fact, all you would need to do is check the primes less than the square root of the number. For example, if you wanted to know if 187 was prime, you would see if it was divisible by 2, 3, 5, 7, 11, and 13 since $\sqrt{187} = 13.67$. In this case, 187 is divisible by 11 ($187 = 11 \cdot 17$). On the other hand, 197 is prime, and all you need to do to verify this is show it is not divisible by the six numbers 2, 3, 5, 7, 11, and 13, since $\sqrt{197} = 14.03$.

Now we are ready for Euclid's Theorem.

Theorem 4: There are an infinite number of prime numbers.

Proof: Suppose there were a finite number of prime numbers. Then the list of them would look like 2, 3, 5, ..., 29, ... , M, where M would be the biggest prime number. Now consider the number you would get if you multiplied all those prime numbers together, and then added 1. Call this number P. In notation,

$$P = (2 \bullet 3 \bullet 5 \bullet 7 \bullet ...... \bullet M) + 1.$$

Now P is certainly bigger than M, and since M is the biggest prime, that means P is not prime. But that means that it must be divisible by some prime number other than itself and 1. However, if you divide P by 2 or 3 or 5, or any prime up to M, the result will always be a remainder of 1. That means it is not divisible by any prime number. Thus we have a contradiction: P must be divisible by a prime number, but P is not divisible by any prime number! Therefore, our original assumption must be wrong. It has therefore been deduced that there must be more than a finite number of primes.

While induction is inadequate as a method of proof, it is an extremely useful method of discovery. Consider the issue of how mathematicians decide what statements to attempt to prove as theorems. These statements are sometimes the result of induction based on either physical or mathematical observations.

Example 1: The sum of consecutive odd integers starting with 1 is (always) a square.

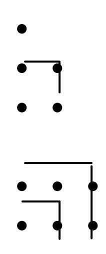
Since this example is less familiar than the previous examples, let's explore how it could have been discovered. Induction might start with the following observations:

$1 = 1^2$

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

and the thought that maybe this pattern continues (forever?). In this case, the sequence of pictures below suggests the theorem, although such pictures do not constitute a formal proof. By the way, these pictures are likely what the words of this example would have suggested to the Greeks. Exponents are a relatively new part of mathematics, so “square of 3” would have meant a 3 by 3 square shape, not 3^2.

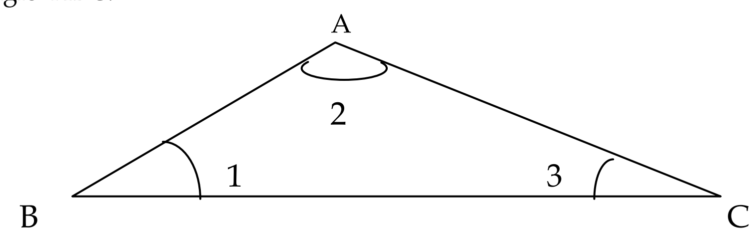


It turns out that the statement in this example is true. The technique used to prove it is one we are not going to study. (It has the somewhat confusing name “mathematical induction”, but is really a way to deduce conclusions.)

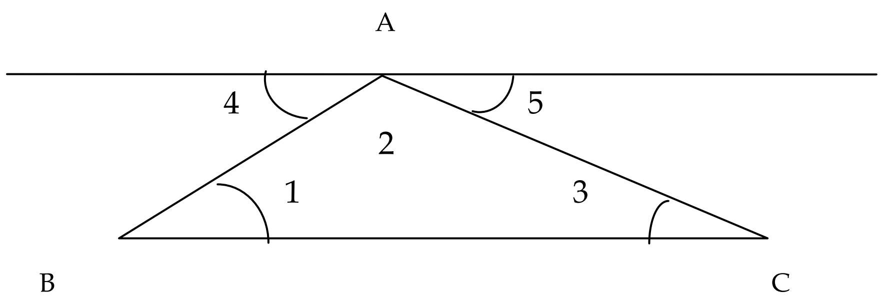
Once a mathematician has decided on a statement which appears to be a theorem, the next issue is how to decide how to do the proof. Sometimes you just can "combine" axioms, and the proof flows fairly smoothly from beginning to end. Most of the time, however, what is required is some creative combination of axioms, especially those which allow for the introduction of some new entity. The following is an example of just such a proof.

Theorem 5: The sum of the interior angles of a triangle is (always) 180°.

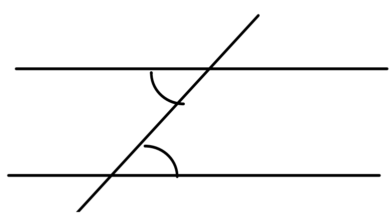
Proof: The diagram below will serve merely as a guide; this is not a proof by picture. Consider a triangle ABC.



The first step in the proof is as creative a step as you can imagine. As an idea, it seems to come from nowhere, perhaps much like the tune running through a composer's head before she turns it into a finished piece of music. Draw a line at point A parallel to BC. (One of Euclid's axioms says you can do this, as we shall see later.) Now the picture takes on a whole new look.



A basic theorem of Euclidean geometry says that when a line cuts through two parallel lines, the so-called alternate interior angles are the same. ("Congruent" is the official term; we will discuss it later.) Such angles are indicated in the diagram below.



In our proof, the theorem about alternate interior angles implies that angles 1 and 4 are the same, and angles 3 and 5 are the same. Since a straight angle is 180°, substitution of equals yields the following result:

180° = angle 4 + angle 2 + angle 5
= angle 1 + angle 2 + angle 3
= sum of the angles of the triangle.

CHAPTER 4: Mathematical Proof **Homework**

1. Which numbers are divisible by 7?
   21, 57, 77, 133, 236, 1344

2. Which numbers are divisible by 8?
   58, 104, 146, 200, 232, 359, 360

3. Which numbers are prime?
   57, 68, 71, 103, 125, 173, 203

4. For centuries, people believed that the formula $n^2 - n + 41$ would yield a prime for every value of n. For n = 1, the formula yields 41, and 41 is prime. For n = 2, the formula yields 43 which is prime.
   a) Verify that for n = 3 and n = 4, the formula $n^2 - n + 41$ yields primes.
   b) Now try it for n = 20. (You may believe me that it works for n = 5, 6, ..., 19.) That is, verify that $20^2 - 20 + 41$ is a prime. Are you becoming convinced?
   c) Now try the formula for n = 40. (Believe me, it works for n = 21, 22, ...39.) That is, verify that $40^2 - 40 + 41$ is a prime. Are you now absolutely convinced?? Something that happens 40 times in a row would seem to always work, right?
   d) Now think about n = 41. By substitution, you get $41^2 - 41 + 41 = 41^2$, which is clearly not prime. What would you conclude about "proof" by induction?

5. A triangle contains angles of 35^{o} and 75^{o}. What is the third angle?

6. A triangle contains angles of 50^{o} and 100^{o}. What is the third angle?

7. A right triangle contains a 25^{o} angle. What is the third angle?

8. How is proving a theorem in mathematics similar or dissimilar from proving today that Jesus rose from the dead?

9. How is proving a theorem in mathematics similar or dissimilar from proving that smoking cigarettes causes cancer?

10. In Acts 1:3, Luke refers to "many convincing proofs." How were these proofs similar or dissimilar from proving mathematical theorems?

11. In Acts 17:31, Paul refers to a proof given by God. How is this proof similar or dissimilar to a mathematical proof?

Selected Answers:

1. 21, 77, 133, 1344
3. 71, 103, 173
4. a) 47 and 53 are prime b) 421 is a prime
5. 70^{o}
7. 65^{o}