

## CHAPTER 5: Rational Numbers

... there are two kinds of people one can call reasonable, those who serve God with all their heart because they know Him, and those who seek Him with all their heart because they do not know Him.<sup>23</sup>

### Counting Numbers

The most basic numbers we use are called the **counting numbers**, or whole numbers: 1, 2, 3, . . . Many cultures have developed words and symbols to represent quite a few of these numbers. The distinction between a number as an abstract idea and a numeral or symbol for the number is an important one. In Roman numerals, "V" represents the same number as our "5", and "X" represents the same number as "10". The way a culture represents numbers is a significant aspect of that culture. We will discuss different ways of representing numbers later.

We have already mentioned the Pythagorean theorem. It is named after a Greek mathematician who lived about 500 B.C. That's about the time the Jewish exiles were returning to Jerusalem. Pythagoras and his followers were early believers in "integration of faith and learning", although in their case, "faith" was not faith in the God of the Bible. But they did believe that religion, philosophy, mathematics, music, and science all fit together into one grand scheme. Among their religious doctrines were a belief that the soul is adversely affected by the body and a belief in reincarnation. Plato was one of the followers of Pythagoras.

At the center of their understanding of the cosmos were the counting numbers. The Pythagorean slogan was "everything is [a whole] number". Without getting into the details, suffice it to say their understanding was quite intricate and impressive. The whole numbers were given mystical meanings. For instance, "four" was identified with the concept of "justice" because it is the product of "equals" ( $2 \cdot 2$ ). The Pythagoreans are important to mathematics per se because they were the first people to study the properties of the counting numbers.

The counting numbers start with 1. Zero as a number is, historically speaking, a much later addition to mathematics. Consider the following: If I show you my hand and you see a dime and a nickel in it, and I ask you how many coins I am holding, you would say "two (2)". However if I drop the coins and show you an empty hand and ask you how many coins I am holding, what would you answer? Most of us would say, "none". Not many people would say, "zero". So what's my point? "Two" and "zero" are number words, but "none" isn't. Zero as a number is a much more advanced concept than two as a number. We will discuss zero more when we later discuss the first culture to understand and use it, the Hindus. For the moment we simply note a few illustrative facts, and one question:

$$\text{Facts:} \quad 3 + 0 = 3 \quad 3 - 0 = 3 \quad 3 \cdot 0 = 0 \quad 0 \div 3 = 0$$

$$\text{Question:} \quad \frac{3}{0} = ?$$

### Rational Numbers

Historically, the next category of numbers to be understood were the so-called **rational numbers**, more commonly called fractions. There are two reasons these numbers are called "rational":

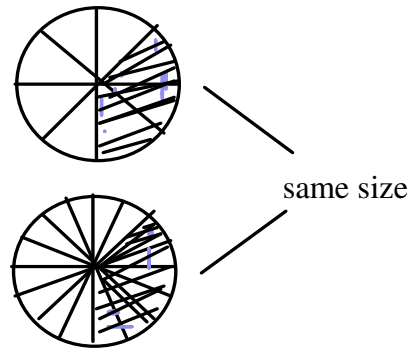
- 1) each is a ratio of whole numbers, and
- 2) they were considered as reasonable, hence meaningful, to the Greeks.

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<sup>23</sup>Pascal, Pensees, p. 74.

Basic to an understanding of rational numbers is the fact that fractions which look different can have the same "value", hence be considered equal. For instance,  $\frac{3}{8}$  is "three one-eighths", and is equal to  $\frac{6}{16}$  or "six one-sixteenths".

Think of two pies, one subdivided into 8 equal pieces, the other subdivided into 16 equal pieces. Three pieces from the first pie is the same amount as 6 pieces from the second.



(  $\frac{3}{8} = \frac{6}{16}$  may seem perfectly obvious to readers who have worked extensively with fractions, but would be less obvious if fractions were new to you. )

### Adding Fractions

Once you believe that rational numbers are meaningful, it would seem useful to consider how to do arithmetic with them. First comes addition. What we need is a procedure that works for any two fractions. But first we need to agree on a principle: our rules will be formulated to fit certain aspects of our experience. That is, these are not some arbitrary rules made up by mathematicians to make life complicated. They are formulated to reflect what really happens in certain settings in everyday life.

Think about it this way: consider a pie cut into eight equal pieces. In this setting it makes sense that three pieces of pie and one more piece of pie would make four pieces of pie, which, by the way, is the same as one half of the pie. That is, "three one-eighths plus one one-eighth equals four one-eighths". So fractions add like this:

$$\frac{3}{8} + \frac{1}{8} = \frac{4}{8} .$$

The rule derived from this observation is that if the numbers on the bottom of the fractions are the same, you simply add the top numbers and keep the bottom number the same. If we simply stated this rule this way without a context, it might sound quite arbitrary. With a context provided, however, it seems quite reasonable.

Now that we have justified the standard rule for adding fractions, it might be useful to look at a type of experience which does not fit this rule. Consider a story about pencils in boxes.

If one box contains 8 pencils, three of which are red, and a second box contains 8 pencils, one of which is red, and you "add" the two boxes together, you have 16 pencils, 4 of which are red. Does this mean we should write:

$$\frac{3}{8} + \frac{1}{8} = \frac{4}{16} ?$$

Or, consider the baseball concept of a batting average. Suppose a player is 1 for 4 ( one hit in four times at bat) in one game, and 2 for 4 in a second game. The player's average was .250 for the first game, and .500 for the second game, but .375 (3 for 8) for the two games combined. In fraction form, we could "add" his averages, and get:

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{8} .$$

Returning to the boxes of pencils, notice that the issue is not that "adding" the boxes together doesn't give 4 red pencils in a box of 16. The issue is that this process is not what adding fractions is all about. Similarly, normal adding of fractions is not designed to do batting averages. Addition of fractions is simply abstracted out of a different type of experience, like cutting pies into pieces. The pencils in the boxes and batting averages are "combined" (or some other term we could agree to use), not "added".

Let's expand the rules for adding fractions. Since two fractions need not have the same number on the bottom (the "denominator"), we would need to create this situation before using the above rule. That is, we need a "common denominator"

**Example 1:**

$$\frac{1}{4} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8} = \frac{5}{8} .$$

If you consider one piece of pie from a pie cut into 4 equal slices equal to 2 pieces from a pie cut into 8 equal slices, then the process makes sense.

We can generalize the above process to add any two fractions, no matter how different their denominators are. Using the analogy of the pie, consider the following example. Add  $\frac{1}{2} + \frac{3}{5}$ . One pie is cut in half, and the other is cut into 5 equal pieces, of which we take 3. How much pie do we have? Consider the following approach. Cut each of the two pieces in the first pie into 5 equal pieces, and cut each of the 5 pieces of the second pie into 2 equal pieces. Now each pie is cut into 10 pieces. In the first pie, we take 5 pieces ( $\frac{1}{2}$ ), and in the second pie we take 6 pieces ( $\frac{3}{5}$ ). All together, we have 11 pieces. Here's how that looks in mathematical notation.

**Example 2:**

$$\frac{1}{2} + \frac{3}{5} = \frac{5}{10} + \frac{6}{10} = \frac{5+6}{10} = \frac{11}{10} .$$

So what is the rule that reflects this process? You can always add two fractions by multiplying the two denominators together to get a common denominator. In the above example, the common denominator was  $2 \cdot 5 = 10$ . Once the common denominator is found, each fraction is written in a form using that denominator by multiplying the numerator and denominator of the fraction by the denominator of the other fraction. Then the fractions can be added as before.

**Examples 3:**

$$\frac{3}{8} + \frac{2}{5} = \frac{3 \cdot 5}{8 \cdot 5} + \frac{2 \cdot 8}{5 \cdot 8} = \frac{15}{40} + \frac{16}{40} = \frac{16+15}{40} = \frac{31}{40} .$$

$$\frac{2}{15} + \frac{3}{10} = \frac{2 \cdot 10}{15 \cdot 10} + \frac{3 \cdot 15}{10 \cdot 15} = \frac{20}{150} + \frac{45}{150} = \frac{20+45}{150} = \frac{65}{150} = \frac{13}{30} .$$

The last example illustrates what happens when the common denominator is not the "least common denominator": the fraction you get when you do the addition is not in lowest terms, i.e., it can be reduced. If we were planning to do a lot of addition of fractions, it would be worth learning how to find this "least common denominator" in general. (If you remember, feel free to use it.) But since the process involved uses topics we have not discussed, we will not bother.

**Percent**

A special kind of fraction is called "percent". The word "percent" is derived from the Latin phrase, *per centum*, "of the hundred". A percent is really a fraction with a denominator of 100. So  $25\% = \frac{25}{100} = .25$

and  $80\% = \frac{80}{100} = .80$ . Special care must be taken when switching between the "%" form and the decimal form of the number. For instance, the decimal .007 is .7%. Occasionally, a percent could be larger than 100%, as in, "The cost of a college education has increased by 350% in the last 15 years."

Batting averages are really percents. A batting average of .300 means that the batter has gotten a hit 30% of the time. Consequently, the comments made above about "adding" of batting averages have analogous forms as statements about percents. In general, it is not appropriate to simply "add" percents. For instance, if a batter has an average of .300 for the first half of the season, and an average of .200 for the second half of the season, we would not want to suggest that her average for the whole season would be  $.300 + .200 = .500$ . Stated in terms of percents, getting a hit 30% of the time in the first half and 20% of the time in the second half should not be combined as  $30\% + 20\% = 50\%$  for the whole season.

Now if you are an avid baseball fan, you probably knew that batting averages don't work the way described above. But if you've not followed baseball, the example may not have been very helpful. And that would be exactly the time that making a mistake would be the most tempting: when the situation is unfamiliar, and your intuition isn't a good guide for which of the rules of math are applicable to the situation.

Let me illustrate how easy it is to make a mistake by telling you the following true story. I once attended a faculty meeting at another college where I was teaching. Enrollment had declined that year, and the president of the school was explaining the decline and its implications to the faculty. I don't remember the exact numbers, but his report went something like this: Male enrollment declined 5%; female enrollment declined 3%; and total enrollment declined 8%! Since an 8% enrollment decline translated into twice as much of a pay cut as a 4% decline, you can understand why the faculty was concerned about the president's math!

What would the correct analysis have been? Suppose last year the school had 1000 males and 1200 females. A 5% male decline would mean a loss of 50 male students; a 3% female decline would mean the loss of 36 female students. This means a total loss of 86 students from a total enrollment last year of 2200.  $\frac{86}{2200} = 3.9\%$ . Combining percents in this case was like combining pencils in boxes above.

The percent problems that you typically face in life are usually one of the following three types. I don't know how you were taught to handle these problems in school, but I would like to present what I think is a very simple approach. In fact, there is nothing to learn as you simply translate the problem from English into the language of algebra.

Example #1: What number is 12% of 54?

$$x = 0.12(54)$$

$$x = 6.48$$

Example #2: 3.2 is 16% of what number?

$$3.2 = 0.16x$$

$$x = 20$$

Example #3: 63 is what percent of 105?

$$63 = x(105)$$

$$x = .6 = 60\%$$

Example #4: Mr. Williams received an unexpected gift of \$16,000. If he gave \$3600 of the \$16,000 to his church, what percent of the check did he give to his church?

This problem is really asking what percent of \$16000 is \$3600?

$$x(16000) = 3600$$

$$x = .225 = 22.5\%$$

## Multiplying Fractions

Multiplication of fractions is actually simpler to do than addition of fractions. To multiply two fractions, you simply multiply the tops together and multiply the bottoms together. That's the rule. But understanding why it works that way is another matter.

Think about multiplication by a whole number. Multiplication is simply a short-cut when you are doing repeated addition of the same number.  $3 \cdot 4$  means "add three 4's", but we just memorize the answer, rather than adding the three 4's each time. In the same way, you would multiply a whole number times a fraction.

$$\text{Example 4: } 3 \cdot \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1+1+1}{8} = \frac{3 \cdot 1}{8} = \frac{3}{8}$$

Multiplying by a fraction is a bit more problematic. Suppose you had a pie cut into eight equal pieces. You took three of the pieces, and decided to give half of what you had to a friend. How much of the original pie would your friend get?

You could imagine that the pie were cut into 16 equal pieces by cutting each of the original pieces in half. You would have six one-sixteenths, and then giving half to your friend would mean giving three one-sixteenths. Dividing what you had into two parts is similar to multiplying by  $1/2$ . To "cut"  $\frac{3}{8}$  of a pie in half would look like this:

$$\frac{1}{2} \cdot \frac{3}{8} = \frac{1}{2} \cdot \frac{6}{16} = \left(\frac{1}{2} \cdot 6\right) \cdot \frac{1}{16} = 3 \cdot \frac{1}{16} = \frac{3}{16}$$

## Dividing Fractions

How about division of fractions? First, think about division of a whole number by a whole number: If you "divide" 6 pies among 3 people, how many pies does each one get? Symbolically, this is  $\frac{6}{3} = 6 \div 3$ , and the answer is 2 pies. Now change the first whole number to a fraction. If you "divide"  $\frac{3}{8}$  of a pie among 2 people, how much does each one get?

$$\text{Example 5: } \frac{\frac{3}{8}}{2} = \frac{3}{8} \div 2 = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}$$

This analysis of a real-life situation suggests the familiar rule for division, often expressed as "invert and multiply". Again, this is not an arbitrary rule; it is designed to match our experience.

Division by a fraction is even more problematic to imagine. It's hard to tell a story in which you share anything with half a person! So we need to change things a bit. If you divide 50 coins into piles of 10 coins each, how many piles will you have? 5 is correct. If you cut a pie into pieces which are one-eighth each, how many pieces will you have? 8 is correct. If you cut 2 pies into eighths, how many pieces do you get? 16, and a rule has emerged.

$$\text{Example 6: } \frac{2}{\frac{1}{8}} = 2 \div \frac{1}{8} = 2 \cdot 8 = 16$$

"Invert and multiply" works again!

More generally, a fraction divided by a fraction works the same way:

$$\text{Example 7: } \frac{\frac{3}{2}}{\frac{5}{8}} = \frac{3}{2} \div \frac{5}{8} = \frac{3}{2} \cdot \frac{8}{5} = \frac{24}{10} = \frac{12}{5} .$$

## Decimals

Now in these days of calculators, many people hardly ever encounter fractions except in cookbook recipes and math classes. Calculators make it much more convenient to change fractions into decimals and then work with decimal notation. The change from fraction to decimal happens by division and is dependent on our base ten positional notation.

$$\text{Example 8: } \frac{3}{8} = .375 = \frac{375}{1000} = 3\left(\frac{1}{10}\right) + 7\left(\frac{1}{100}\right) + 5\left(\frac{1}{1000}\right) .$$

Many rational numbers are like  $\frac{3}{8}$  in that they can be expressed exactly as a “terminating” decimal. However, there are other rational numbers whose decimal form is “non-terminating”, i.e., never-ending. These rational numbers have a “repeating” decimal. For instance,  $\frac{1}{3} = .333$  in which the “3” repeats forever. The number you would see on a calculator, like .33333333, is merely a good approximation to  $\frac{1}{3}$ . Some “repeating” decimals have a pattern of digits that repeats, not just a single digit. An example of this is  $\frac{2}{7}$ . Its decimal is .28571428571428..., which has a six-digit repeating pattern. Such numbers are often written with a line over the repeated pattern, e.g.,  $\frac{1}{3} = .\overline{3}$  and  $\frac{2}{7} = \overline{.285714}$ .

This distinction between “terminating” and “repeating” may seem fairly straight-forward. We shall see, however, which category a rational number belongs in, “terminating” or “repeating”, depends on what base you are using to write numbers. Everything we have said so far is about decimals, i.e., base 10. Later we shall see what can happen if you write numbers in other bases.

Here’s another somewhat strange example. Consider the repeating decimal .9999..... What rational number does it represent? I claim it is the number 1. We will learn two general procedures to justify this later. For the moment, the following ad hoc procedure will do

$$.9999\dots = 3 \cdot (.333\dots) = 3 \cdot \frac{1}{3} = 1.$$

One last comment. We have rather naively been think of repeating decimals as ones which have a pattern of digits that repeat forever. What exactly does “forever” mean here? Can we really know what happens “forever” in the decimal form of  $\frac{2}{7}$ ? Is it really possible to multiply an infinitely long string of 3’s by 3 and get an “infinitely long string of 9’s, like we did in  $3 \cdot (.333\dots) = .9999\dots$  above? We shall have much more to say about “infinity”.

## CHAPTER 5: Rational numbers

## Homework

1. Calculate:

a.  $\frac{3}{7} + \frac{2}{7}$

b.  $\frac{1}{2} + \frac{1}{4}$

c.  $\frac{2}{3} \cdot 4$

d.  $\frac{\frac{4}{9}}{2}$

e.  $\frac{2}{3} \div 3$  (division)

f.  $\frac{4}{5} \cdot \frac{1}{2}$

g.  $\frac{5}{2} + \frac{2}{5}$

h.  $6 \div \frac{3}{4}$  (division)

i.  $0 \div \frac{2}{3}$  (division)

j.  $\frac{\frac{6}{25}}{\frac{2}{5}}$

k.  $\frac{5}{6} \cdot 0$

l.  $\frac{3}{8} + \frac{1}{6}$

m.  $\frac{3}{4} \div 0$  (division)

n.  $\frac{2}{3} - \frac{1}{9}$

o.  $\frac{3}{5} - \frac{1}{3}$

p.  $4 \div \frac{3}{5}$  (division)

2. Calculate:

a.  $\frac{1}{2} + \frac{3}{4} + \frac{1}{8}$

b.  $\frac{\frac{3}{7}}{4}$

c.  $\frac{4}{5} \cdot \frac{15}{16}$

d.  $\frac{1}{3} + \frac{1}{6} + \frac{1}{2}$

e.  $\frac{1}{3} + \frac{1}{2} - \frac{1}{6}$

f.  $\frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{3}$

$$g. \quad \frac{\frac{4}{24}}{\frac{2}{7}}$$

$$h. \quad 5 \div \frac{6}{7} \text{ (division)}$$

$$i. \quad \left(\frac{1}{3} + \frac{1}{2}\right) \cdot \frac{1}{2}$$

$$j. \quad \left(\frac{2}{5} + \frac{3}{8}\right) \cdot 0$$

$$k. \quad \left(\frac{1}{3} + \frac{1}{2}\right) \div 2$$

$$l. \quad 4 \cdot \left(\frac{5}{2} + \frac{3}{4}\right)$$

$$m. \quad \left(\frac{1}{3} + \frac{1}{2}\right) \div \frac{1}{2}$$

$$n. \quad \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)$$

3. Write the fraction either: 1) exactly as a terminating decimal; or 2) by showing the repeating pattern.

$$a. \quad \frac{7}{10}$$

$$b. \quad \frac{3}{20}$$

$$c. \quad \frac{2}{3}$$

$$d. \quad \frac{3}{5}$$

$$e. \quad \frac{22}{7}$$

$$f. \quad \frac{3}{8}$$

$$g. \quad \frac{7}{9}$$

$$h. \quad \frac{3}{11}$$

4. Solve each of the following percent problems.

a. What number is 22% of 14.25?

b. 279 is 90% of what number?

c. What percent of 64 is 8?

d. .54 is 150% of what number?

e. 52 is what percent of 4?

f. What number is  $\frac{2}{3}\%$  of 711?

g. Mom and Dad shared driving responsibilities on a 500-mile trip. What percent of the 500 miles did Dad drive if he drove 259 miles?

h. A young man rents a room for \$420 per month. Find his monthly income if 20% of his income is spent on rent.

i. During the school year, Sharlene was present 165 days and absent 15 days. What percent of the school year was Sharlene absent?

j. What is the amount of a 20% discount on a \$25 item?

k. To date, Biola's basketball team has won 18 games and lost 6. What percent of the games played have they won?

l. Some granola is 10% fat. How much granola would you have to eat in order to consume 4 ounces of fat?



## Selected Answers:

1. a.  $\frac{5}{7}$     b.  $\frac{3}{4}$     d.  $\frac{2}{9}$     e.  $\frac{2}{9}$     f.  $\frac{4}{10} = \frac{2}{5}$
- h. 8    i. 0    k. 0    l.  $\frac{13}{24}$     m. can't be done
2. a.  $\frac{11}{8}$     c.  $\frac{3}{4}$     e.  $\frac{2}{3}$     f.  $\frac{5}{28}$     g.  $\frac{7}{12}$
- h.  $\frac{35}{6}$     i.  $\frac{5}{12}$     k.  $\frac{5}{12}$
3. a. .7    b. .15    c.  $\overline{.6}$     d. .6    e.  $\overline{3.142857}$
4. a. 3.135    b. 310    c. 12.5%    f. 4.74    g. 51.8%    h. \$2100
- k. 75%    l. 40 ounces or 2.5 lbs.