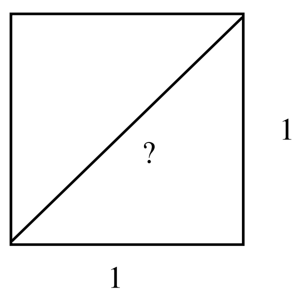
CHAPTER 6: Irrational and Negative Numbers

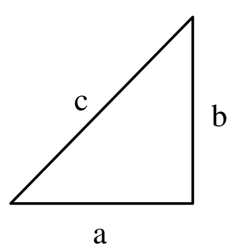
> The prophecies, the very miracles and proofs of our religion, are not of such a nature that they can be said to be absolutely convincing. But they are also of such a kind that it cannot be said that it is unreasonable to believe them.[24]

As discussed in the previous chapter, the Pythagoreans held as a foundational principle that "everything is [a whole] number". For them, this meant that whole numbers were the essence of all reality. Since rational numbers were ratios of whole numbers, they were also legitimate numbers. For instance, in music, tones and length of vibrating strings were related by whole numbers. Based on this principle, the Pythagoreans developed the first rational and mathematical philosophy of nature. In this way, they were the forerunners of the modern scientists. It wasn't long, however, until a crisis arose in their philosophy.

Consider the following simple geometry problem. What is the length of the diagonal of a 1 by 1 square?



The Pythagorean theorem states that in a right triangle, $a^2 + b^2 = c^2$:



So the length of the diagonal above is $c = \sqrt{2}$. For us, writing this result is a rather casual piece of work. But " $\sqrt{2}$ " is just a symbol, and the Pythagoreans had no such symbol. But even if they had, the real question was: "What rational number is this?" This may seem like an innocent enough question. But for the Pythagoreans, its answer had far-reaching implications. The proof of the following theorem is another example of an indirect proof, where we begin the proof by assuming what we want to prove is false.

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[24]Pascal, Pensees, p. 187.

Theorem: $\sqrt{2}$ is not a rational number.

Proof: Suppose $\sqrt{2}$ were a rational number. Then $\sqrt{2} = \frac{a}{b}$ for some whole numbers a and b, and we can assume this fraction is written in lowest terms. Then $\frac{a^2}{b^2} = 2$, or $a^2 = 2b^2$ by multiplying both sides of the equation by b^2. Now $2b^2$ is clearly even, so a^2 is even, and hence the number a is even by a theorem we proved earlier (see Chapter 4). So, $a = 2n$ for some whole number n. Then by substitution, $a^2 = (2n)^2 = 4n^2 = 2b^2$, which implies $2n^2 = b^2$. So b^2 is even, and hence the number b is even. But this means we have found out that both a and b are even. Thus the fraction $\frac{a}{b}$ isn't in lowest terms. That's a contradiction of an earlier statement. So our original assumption, that $\sqrt{2}$ is rational, cannot be true. So $\sqrt{2}$ is not a rational number.

Now, for the Pythagoreans, saying that a number was not rational was essentially saying that it wasn't really a number at all! Rational numbers were ratios of whole numbers, and so they were real. If everything was made up of whole numbers, there simply was no room in their universe for numbers that were not rational! This theorem pointed to a major flaw in their philosophy, their science, and their religion!

Even Greeks who were not Pythagoreans found such numbers to be problematic. These "numbers" were dubbed "irrational" numbers, both because they could not be written as a ratio of whole numbers, and also because they didn't make sense in the way that the rational numbers did. The Greeks never figured out an effective way to do arithmetic with irrational numbers.

Morris Kline has commented, "The mathematician's refusal over centuries to grant irrational numbers the status of numbers illustrates one of the surprising features of the history of mathematics. New ideas are often as unacceptable in this field as they are in politics, religion, and economics."[25] Actually, new ideas have often been controversial in mathematics. We will see several more examples as this course develops.

Arithmetic with Square Roots

Let's start with addition. Let’s take an exploratory approach. Is what follows the way it would work?

$$\sqrt{2} + \sqrt{3} = \sqrt{5} \quad ?$$

Without using a calculator, you may not know how to tell whether this is correct. Try numbers that are easier:

$$\sqrt{4} + \sqrt{9} = \sqrt{13} \quad ?$$

NO, this isn't correct. Since 2 + 3 = 5, the right answer is

$$\sqrt{4} + \sqrt{9} = \sqrt{25} \quad !$$

But how in the world do you get the 25 from 4 and 9? The Hindus and Arabs discovered the answer, and we will look at their solution later. For the moment, we will adopt the typical high school algebra approach, and say that we "can't do" $\sqrt{2} + \sqrt{3}$. What we mean is that we can't write a single

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[25]Morris Kline, Mathematics for the Nonmathematician, p. 69.

square root expression which is exactly equal to this number. With a calculator, we could easily get an approximate answer, but that is not the real issue at the moment.

Now let's try multiplication. Once again we try:

$\sqrt{2} \bullet \sqrt{3} = \sqrt{6}$?

This time working with easier numbers leads to

$\sqrt{4} \bullet \sqrt{9} = 2 \bullet 3 = 6 = \sqrt{36} = \sqrt{4 \bullet 9}$ YES! This works. It looks like we have good reason to make the following :

Rule: $\sqrt{a} \bullet \sqrt{b} = \sqrt{a \bullet b}$

And in fact this rule could be proved, now that we have discovered it. Actually, it is a special case of a more general rule we will encounter later.

We ought to state explicitly what we have now done in several instances. When we want to generalize, we must make sure that the more general rule agrees with earlier rules. For instance, since 2 + 3 = 5, the rule for adding fractions had better work when the whole numbers are viewed as fractions: $\frac{2}{1} + \frac{3}{1} = \frac{5}{1}$. Similarly, the rule for multiplying radicals needed to work when 2, 3 and 6 were viewed as radicals: $\sqrt{4} \bullet \sqrt{9} = \sqrt{36}$.

What about division? As long as we avoid dividing by zero, we can take the above multiplication rule, and divide both sides by $\sqrt{b}$ to get

$\sqrt{a} = \frac{\sqrt{a \bullet b}}{\sqrt{b}}$. But $\sqrt{a}$ is also seen to be equal to $\sqrt{\frac{a \bullet b}{b}}$ by canceling the b's.

So $\frac{\sqrt{a \bullet b}}{\sqrt{b}} = \sqrt{\frac{a \bullet b}{b}}$. Let m = a•b and n = b. Then we have derived the following rule for division:

Rule: $\frac{\sqrt{m}}{\sqrt{n}} = \sqrt{\frac{m}{n}}$

For example, $\frac{\sqrt{200}}{\sqrt{8}} = \sqrt{\frac{200}{8}} = \sqrt{25} = 5.$

Negative numbers

Negative numbers were not recognized as legitimate numbers in Western Europe until the Renaissance (around 1300 AD). Actually, outside of mathematics books, negative numbers are fairly uncommon. Perhaps your checkbook balance occasionally dips into the negative number realm, or perhaps you have spent some time where temperatures can be below zero. For the most part though, life could go on quite well without negative numbers, don't you think? Well, mathematics would not.

Negative numbers arose more because of forces internal to mathematics than because of practical needs. As soon as you introduce the operation of subtraction, you are in trouble if you only use positive numbers. For instance, 5 - 3 = 2. But what about 3 - 5? We could just make a rule that says you're not allowed to subtract a bigger number from a smaller number. But that rule would seem to be arbitrary, wouldn't it?

If the temperature was 3 degrees, and it dropped 5 degrees, what would the temperature be now? If you had $3 in your checking account, and you wrote a check for $5, what's your new balance? Even if these applications seem avoidable, the mathematician still wants an answer, just because the question has been posed. If reason can provide us with answers that make sense, then let there be negative numbers.

Here are illustrations of the rules that make consistent sense of operations with negative numbers.

Examples and Rules:

1. Addition involving negative numbers.

   a. $(-2) + (-3) = -5$ $(-a) + (-b) = -(a + b)$

   In this case, think of adding a new debt to an old debt: your total debt increases.

   b. $8 + (-5) = 3$ $a + (-b) = a - b$

   c. $-7 + 13 = 6$

   d. $(-5) + 2 = -3$ $(-a) + b = -(a-b)$

   e. $7 + (-9) = -2$

2. Subtraction involving negative numbers

   a. $6 - (-2) = 8$ $a - (-b) = a + b$

   b. $-7 - 3 = -10$ $-a - b = -(a + b)$

3. Multiplying a negative and a positive.

   a. $3 \bullet (-5) = -15$ $a \bullet (-b) = ab$

   Think of the total of three debts of $5 each.

   b. $(-4) \bullet 7 = -28$

4. Multiplying two negatives.

   a. $(-3) \bullet (-5) = +15$ $(-a) \bullet (-b) = ab$

   This rule is more complicated to explain. Here's an explanation by the indirect approach. If $(-3) \bullet (-5) = -15$, we would have $(-3)(-5) = 3(-5)$, so by cancellation of a "– 5" from both sides of the equation , $-3 = 3$. Since this can't be true, $(-3) \bullet (-5) = +15$ is the only acceptable option. We accept it because the other option simply doesn't make sense.

   If that explanation doesn't satisfy you, I should tell you that there is a more formal, direct mathematical proof. However, it would take more time to present than we want to spend on this issue at the moment.

   b. $(-4) \bullet (-100) = 400$

CHAPTER 6: Irrational and Negative Numbers **Homework**

1. Do the following arithmetic, where possible, according to the given rules.

a. $\sqrt{5} + \sqrt{10} =$ b. $\sqrt{4} + \sqrt{9} =$ c. $\sqrt{3} + \sqrt{3} =$ d. $\sqrt{5} \bullet \sqrt{7} =$

e. $\sqrt{6} \bullet \sqrt{3} =$ f. $\sqrt{2} \bullet \sqrt{2} =$ g. $\frac{\sqrt{20}}{\sqrt{5}} =$ h. $\frac{\sqrt{24}}{\sqrt{8}} =$

i. $\sqrt{5} + \sqrt{5} =$ j. $\frac{\sqrt{30}}{\sqrt{6}} =$ k. $\sqrt{5} + \sqrt{7} =$ l. $\frac{\sqrt{64}}{\sqrt{36}} =$

2. Do the following arithmetic, where possible, according to the given rules.

a. $\sqrt{3} + \sqrt{8} =$ b. $\sqrt{2} + \sqrt{7} =$ c. $\sqrt{7} + \sqrt{7} =$ d. $\sqrt{4} \bullet \sqrt{6} =$

e. $\sqrt{5} \bullet \sqrt{3} =$ f. $\sqrt{3} \bullet \sqrt{3} =$ g. $\frac{\sqrt{20}}{\sqrt{4}} =$ h. $\frac{\sqrt{25}}{\sqrt{5}} =$

i. $\sqrt{8} + \sqrt{8} =$ j. $\sqrt{6} + \sqrt{8} =$ k. $\frac{\sqrt{36}}{\sqrt{64}} =$ L. $\frac{\sqrt{18}}{\sqrt{6}} =$

3. Do the following arithmetic.

a. (– 3) + 5 = b. (– 4) + (– 2) = c. (– 6) + (– 3) = d. 4 + (– 7) =

e. 7 – (– 3) = f. 8 – (– 10) = g. (– 3) • (+4) = h. (– 4) • (– 3) =

i. (– 6) • (– 2) =

4. Do the following arithmetic.

a. (– 4) + 6 +(-2) = b. (– 6) + (– 3) -2 c. (– 5) + (– 2) –(-4) d. 6 + (– 7) -3

e. 7 – (– 6) – (-5) f. 8 – (– 9) -18 g. $\frac{(-4) \bullet (+5)}{-10} =$ h. $\frac{(-4) \bullet (+7)}{-7} =$

i. $\frac{(-7) \bullet (-2)}{-4} =$

5. Do the following arithmetic.

a. –2 [(-3) – (-1)] b. –3 [-2 + (-2)] c. (– 3) (– 2) +(-2) d. -3 [-2-(-2) (-2)]

e. 2 –3 [-3 + 1]

Selected Answers

1. a. can't do anything b. $2+3 = 5$ c. $2\sqrt{3}$ d. $\sqrt{35}$ e. $\sqrt{18} = 3\sqrt{2}$ f. $\sqrt{4} = 2$
g. $\sqrt{4} = 2$ h. $\sqrt{3}$
2. a. can't be done b. can't be done c. $2\sqrt{7}$ d. $\sqrt{24}$ or $2\sqrt{6}$ e. $\sqrt{15}$ f. $\sqrt{9} = 3$

3. a. 2 b. -6 c. -9 d. -3 e. 10
4. a. 0 b. –11 c. –3 d. –4 e. 18
5. a. 4 b. 12 c. 4