CHAPTER 7: Mathematics of India

> Grace is indeed needed to turn a man into a saint; and he who doubts it does not know what a saint or a man is.[26]

Perhaps the most well-known contribution to mathematics by the Hindu cultures are the numerals we use today. The symbols "1", "2", etc. are often referred to as Hindu-Arabic numerals because they had their origin in India.

Mathematics in Hindu culture was motivated mainly by astronomy and astrology. There were no pure mathematics books. The presentation of mathematical results was typically in verse (and mystical language). There are probably two good explanations for this: it adds interest (if you come from a culture that values poetry) and aids the memory.

The geometry of India was empirical, approximate, and without proofs, much like that of Egypt and Babylon. The Greek approach through the axiomatic method was very unique in the ancient world.

In Hindu culture, we find the earliest rudiments of trigonometry. This includes the "sine" concept which we will discuss later. The Hindus used this idea from about the fourth century AD.

Zero

The main area of mathematical development in India was number theory. In arithmetic, algorithms were developed for all of the basic operations, but no proofs were provided. In this regard, the rules for working with zero as a number were developed. By 600 AD, 0 was regarded as a number just like 1,2,3,.... By 800 AD, the rules $0 \bullet 0 = 0$ and $a - 0 = a$ were explicitly stated by the Hindus. However, they were still "in process", because one other rule stated was $\frac{a}{0} = a$. This is simply not true. A division fact is always based on a multiplication fact; for instance, $\frac{6}{3} = 2$ is based on $2 \bullet 3 = 6$. In the same way $\frac{a}{0} = a$ would require $a \bullet 0 = a$ as its basis - but of course $a \bullet 0 = 0$, not a.

By 1200 AD, a different understanding of division by zero had emerged: $\frac{a}{0}$ was regarded as infinite. Infinity is an interesting concept, dealt with differently in different times and places. We will encounter it on a number of occasions throughout the course.

Adding Irrationals

Hindus were the first to use other numbers which were more slowly accepted in western Europe. Negative numbers were used (to represent debts) by 600 AD. Irrational numbers were accepted in a way they had not been accepted by the Greeks. In fact, operations with irrational numbers were developed so that they could be "reckoned like integers". Earlier we concluded that we could not typically add irrational numbers: $\sqrt{2} + \sqrt{3}$ could not be done. The Hindus, however, were more creative. The next two examples may have been the type of inductive evidence that led them to a general rule:

Examples: 1. $1 + 2 = \sqrt{1} + \sqrt{4} = \sqrt{(1+4)+2\sqrt{1 \bullet 4}} = \sqrt{9} = 3$

2. $3 + 4 = \sqrt{9} + \sqrt{16} = \sqrt{9 + 16 + 2\sqrt{9 \bullet 16}} = \sqrt{49} = 7$

The general rule suggested by these two examples is:

Rule: $\sqrt{a} + \sqrt{b} = \sqrt{(a+b)+2\sqrt{ab}}$.

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[26] Pascal, Pensees, p. 167.

Consequently, our earlier problem, which we said previously could not be done, would have the answer,

$$\sqrt{2} + \sqrt{3} = \sqrt{(2+3)+2\sqrt{2\bullet 3}} = \sqrt{5+2\sqrt{6}} .$$

Since we can't check this numerical result exactly (for instance, we can't even find $\sqrt{2}$ exactly), you might wonder whether we really have enough evidence to justify this rule. If you believe calculators, my calculator says

$$\sqrt{2} + \sqrt{3} = 1.414213562 + 1.732050808 = 3.14626437$$

and

$$\sqrt{5+2\sqrt{6}} = \sqrt{5+2(2.449489743)} = \sqrt{9.898979486} = 3.14626437.$$

Here are some additional examples:

Examples: 3. $\sqrt{3} + \sqrt{8} = \sqrt{(3+8)+2\sqrt{3\bullet 8}} = \sqrt{11+2\sqrt{24}}$

$\sqrt{3} + \sqrt{8} = 1.7320508 + 2.8284271 = 4.5604779$

$\sqrt{11+2\sqrt{24}} = \sqrt{11+2\,(4.8989795\qquad)} = \sqrt{20.797959} = 4.5604779$

4. $\sqrt{5} + \sqrt{7} = \sqrt{5+7+2\sqrt{5\bullet 7}} = \sqrt{12+2\sqrt{35}}$

$\sqrt{5} + \sqrt{7} = 2.236068 + 2.6457513 = 4.8818193$

$\sqrt{12+2\sqrt{35}} = \sqrt{12+2\,(5.9160798)} = \sqrt{23.83216} = 4.8818193$

The best reason to believe this rule would be a deductive proof. If you remember some algebra, try squaring both sides of the equation; you should get the same expression on both sides. More formally, the rule can be derived this way:

$$\sqrt{a} + \sqrt{b} = \sqrt{(\sqrt{a}+\sqrt{b})^2} = \sqrt{(\sqrt{a})^2+2\sqrt{a}\sqrt{b}+(\sqrt{b})^2} = \sqrt{a+2\sqrt{ab}+b} = \sqrt{(a+b)+2\sqrt{ab}}$$

Speaking of algebra, Hindu algebra was almost symbolic by about 1000 AD. That is, operations such as addition and multiplication were represented by symbols, as were the unknown numbers. That may not seem like a big deal to you. However, prior to that time, and continuing until about 1500 AD in western Europe, algebra was done almost exclusively in words! Consider this example: find the number which, when multiplied by three and increased by five gives a number which is equal to thirty-two. How would you like to solve this problem by writing an essay?

CHAPTER 7: Mathematics of India **Homework**

Do all the following problems without a calculator first, writing the answer in simplest form. Then compare your results with those of a calculator.

1. $\sqrt{2} + \sqrt{8} =$
2. $\sqrt{13} \bullet 0 =$
3. $\sqrt{4} + \sqrt{16} =$
4. $\sqrt{7} \div 0$ (division) =
5. $0 \div \sqrt{4}$ (division) =
6. $\sqrt{5} + \sqrt{20} =$
7. $\sqrt{7} + \sqrt{11} =$
8. $\sqrt{12} + \sqrt{18} =$
9. $\sqrt{6} \bullet 0 =$
10. $\sqrt{5} + \sqrt{8} =$
11. $\sqrt{10} \div 0$ (division) =
12. $\sqrt{11} + \sqrt{5} =$
13. $0 \div \sqrt{3}$ (division) =
14. $\sqrt{3} + \sqrt{13} =$
15. $\sqrt{7} + \sqrt{17} =$

Selected Answers:

1. $\sqrt{18} = 3\sqrt{2} = 4.242640687$
2. 0
3. 6
4. can’t be done
7. $\sqrt{18+2\sqrt{77}} = 5.962376101$
8. $\sqrt{30 + 2\sqrt{216}} = 7.7067423$
9. 0
10. $\sqrt{13 + 2\sqrt{40}} = 5.0644951$
11. "can't be done " or "infinity"
12. $\sqrt{16 + 2\sqrt{55}} = 5.5526928$