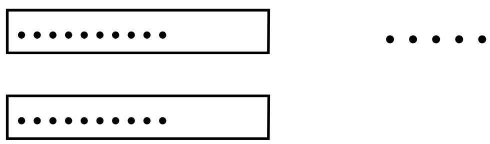
CHAPTER 10: Different Bases (Numeral Systems)

> Do little things as though they were great, because of the majesty of Jesus Christ who does them in us, and who lives our life; and do the greatest things as though they were little and easy, because of His omnipotence.[29]

Positional numeral systems depend on grouping and place-value to write numbers. The base of a system is the size of the group used. In base 10, we organize into groups of ten, (ten groups of ten is one hundred, ten hundred is a thousand; etc.).

Example: 25 represents objects which are being considered as 2 groups of 10, and 5 individuals.

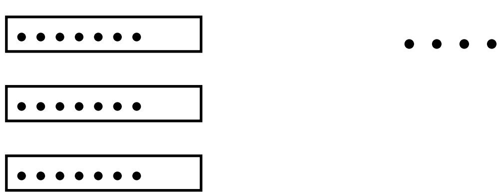


Recall that in base 10, 123 means 1•100 + 2•10 + 3. Words like "hundred" for 100 and "twenty" for 20 are base ten words. As we change to other bases, we will avoid such words. In base 10, we use symbols for 0,1,2,...9. All other numbers are represented by combinations of these ten symbols. Perhaps less familiar is the fact that 1.23 means $1 + 2\cdot\frac{1}{10} + 3\cdot\frac{1}{100}$. "Decimal point" is also a base ten term; in general we'll just call it the "point".

Writing numbers in bases other than 10

Earlier we commented that the Babylonians worked with base 60. We won't use this as an example simply because the numbers involved are so large. Instead, let's see how you would write numbers (remember, they are abstract concepts with many ways of being written) in base 7. We will only need symbols for 0,1,2,3,4,5, and 6. To indicate the base being used (if it is not the usual base 10), we will add a subscript to the numeral. The first column on the right is still the 1's column. The next column is the 7's column. The next column is the 49's column, because 7•7 = 49, just like the third column in base ten is the 100's column (100 = 10•10). If you are looking for a "practical application" of this procedure, think about the way we group 7 days into 1 week.

Example: The 25 dots in the previous example we grouped by 10's. Now we will group by 7's:



So, $25 = 3 \cdot 7 + 4 = 34_7$. In the "application" mentioned above, you could say that 25 days = 3 weeks and 4 days.

Examples in base 7:

10_7 means 1•7 + 0 = 7; notice 10_7 is not to be read "ten".

123_7 means 1•49 + 2•7 + 3, and so is equal to 66.

---

[29]Pascal, Pensees, p. 181.

 71

Let's consider another situation in which a similar notion appears, namely writing lengths in feet and inches. Since 12 inches = 1 foot, we could write 5 feet 7 inches as 5 times 12 , plus 7 inches, or 67 inches. Similarly, 7 feet 2 inches is equal to 86 inches. Now compare those calculations to the following.

Examples in base 12:

$57_{12} = 5 \bullet 12 + 7 = 67$

$72_{12} = 7 \bullet 12 + 2 = 86$

Now let's reverse the process in the context of feet and inches. We can write, for instance, 45 inches as 3 feet 9 inches. How did we do that change? Each "group of 12 inches" was converted to "1 foot". How did we get 3 groups of 12 inches, and 9 inches left over? My guess is that you did division:

$$\begin{array}{r} \textcircled{3} \\ 12\overline{)45} \\ 36 \\ \hline \textcircled{9} \end{array}$$

Example in feet and inches: Convert 55 inches to feet and inches.
Solution: 48 inches = 4 feet, with 55 - 48 = 7 inches remaining. 55 inches = 4 feet 7 inches.

Example in base 12: Write 55 (which is base 10) in base 12.
Solution: Divide 55 by 12. The quotient is 4, and this tells you how many groups of 12 there are in 55. The remainder is 7, which tells you how many of the 55 remain ungrouped. So 55 = 47_{12} .

To count objects in base 10, you might form as many groups of 10 as possible, then put 10 groups of 10 together to form a group of 100, and do that as often as possible, and so on. To count objects in base 7, you would form as many groups of 7 as possible, then put 7 groups of 7 together to form a group of 49, and so on. The mathematical procedure used to write a number in base 7, when the number is given to you in base 10, is the mathematical analog of the counting procedure. Mathematically, we "group" into 7's by dividing by 7. The quotient (how many times 7 divides into the given number) would be the number of groups of 7 you would get. The remainder would be the number of objects left over as individuals after the grouping was complete.

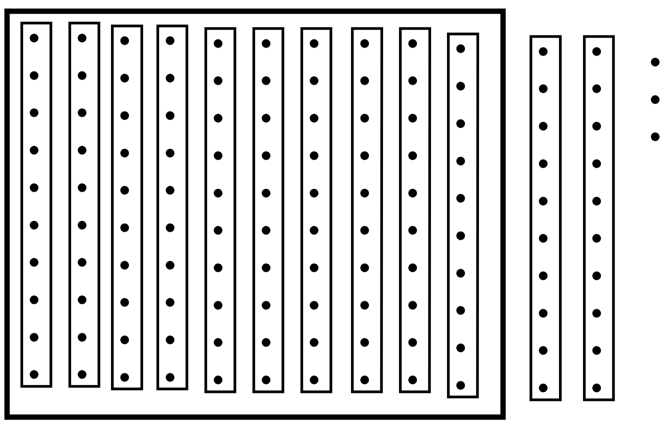
For instance, if we want to write 25 in base 7, we divide 25 by 7. The quotient is 3, and the remainder is 4. So 25 is 3 groups of 7 and 4 more, i.e., 34_7. For larger numbers, the process is a bit more complicated, although founded on the same principles. Successive division by 7 is needed until the quotient is less than 7, indicating that all possible grouping has been accomplished. The answer will consist of the remainders, written beginning on the right, and the final quotient.

Example: Write 123 in base 7.
Solution: Divide 123 by 7. The quotient is 17, with a remainder of 4. Keep the 4. Divide 17 by 7. The quotient is 2, with a remainder of 3. Keep the 3. Since the quotient (2) is now less than 7, we stop. The answer is 234_7. The remainders are written beginning on the right, and the 2 is the final quotient. (See calculations on next page)

$$\begin{array}{r} 17 \\ 7\overline{)123} \\ \underline{7} \\ 53 \\ \underline{49} \\ \textcircled{4} \end{array} \qquad \begin{array}{r} \textcircled{2} \\ 7\overline{)17} \\ \underline{14} \\ \textcircled{3} \end{array}$$

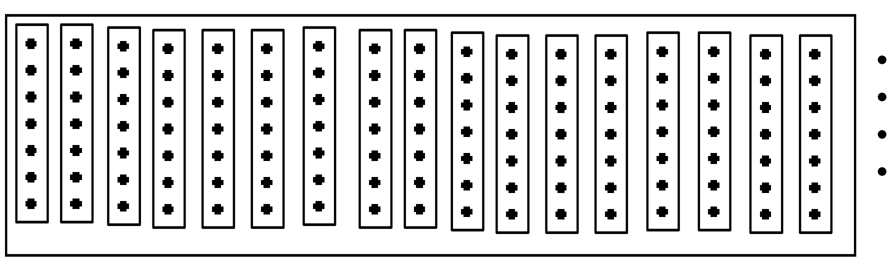
In the above example, dividing 123 by 7 corresponds to putting all 123 objects into groups of 7. There would be 17 groups of 7, with 4 objects left over. Then the 17 groups of 7 objects would be grouped by 7's, forming 2 groups of 7 groups of 7 (2 groups of 49) with 3 groups of 7 left over.

123 (in base ten):

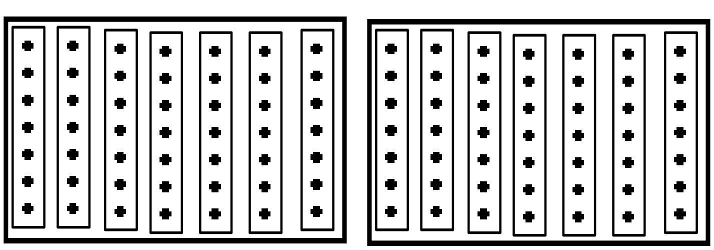


Converting to base 7:

First division:



Second division:



Example: Write 546 in base 7.

Solution: Divide 546 by 7. The quotient is 78, with a remainder of 0. Keep the 0. Divide 78 by 7. The quotient is 11, with a remainder of 1. Keep the 1. Divide 11 by 7. The quotient is 1, with a remainder of 4. Keep the 4. Since the quotient (1) is less than 7, we stop. The answer is 1410_7.

Check: $1410_7 = 1 \cdot (7 \cdot 7 \cdot 7) + 4 \cdot (7 \cdot 7) + 1 \cdot 7 + 0$
$= 343 + 196 + 7$
$= 546$

Example: Write 909 in base 7.

Solution:

```
   129           18          (2)
7)909         7)129        7)18
  7             7            14
  20            59           (4)
  14            56
   69           (3)
   63
   (6)
```

So $909 = 2436_7$.

The same procedures apply to other bases. For instance, the columns in base 5 would be, from the right, the 1's column, the 5's column, the 25's column, the 125's column, etc. Only the symbols 0,1,2,3, and 4 would be used. Here are some examples to illustrate problems similar to the those above.

Examples in base 5

$321_5 = 3 \cdot 25 + 2 \cdot 5 + 1 = 75 + 10 + 1 = 86$

$2403_5 = 2 \cdot 125 + 4 \cdot 25 + 0 \cdot 5 + 3 = 250 + 100 + 0 + 3 = 353$

Example: Write 132 in base 5.

Solution: Divide 132 by 5. The quotient is 26, with a remainder of 2. Keep the 2. Divide 26 by 5. The quotient is 5, with a remainder of 1. Keep the 1. Divide 5 by 5. The quotient is 1 with a remainder of 0. Since the quotient (1) is now less than 5, we stop. The answer is 1012_5. The remainders are written beginning on the right, and the 1 is the final quotient.

Example: Write 432 in base 5.

Solution:

$$\begin{array}{r} 86 \\ 5\overline{)432} \\ \underline{40} \\ 32 \\ \underline{30} \\ \textcircled{2} \end{array} \qquad \begin{array}{r} 17 \\ 5\overline{)86} \\ \underline{5} \\ 36 \\ \underline{35} \\ \textcircled{1} \end{array} \qquad \begin{array}{r} \textcircled{3} \\ 5\overline{)17} \\ \underline{15} \\ \textcircled{2} \end{array}$$

So $432 = 3212_5$.

Check: $3212_5 = 3 \bullet 125 + 2 \bullet 25 + 1 \bullet 5 + 2$
$= 375 + 50 + 5 + 2$
$= 432$

Addition in different bases

Now I'm interested in the problem of addition of numbers. First let's remind ourselves about how this works in base 10. When we add numbers like 24 and 63, we add the numbers in the one's column: 4 + 3 = 7, and the numbers in the 10's column: 2 + 6 = 8, and write the answer: 87. If, however, the numbers we wanted to add were 37 and 98, the process is a bit more complicated. We would add the numbers in the one's column: 7 + 8 = 15. We would add the numbers in the 10's column: 3 + 9 = 12. But we wouldn't write the answer as 1215. Returning to the one's column, we would think of 15 as 1 group of 10, and 5 ones. So the 1 would be "carried over" into the ten's column. So actually; in the ten's column we would have 1 + 3 + 9 = 13. The correct thinking at this point is that the "1" in "13" must be "carried over" into the next column, the hundred's, so we get 135 as the answer.

As an "application" let's return to our analogy about feet and inches. Suppose you had a board that was 3 feet 9 inches long, and another board that was 2 feet 7 inches long. If you laid them end-to-end, how long would they be? For emphasis I'll write a two-step solution, although my guess is that many of you would simply write the answer because you are so accustomed to doing this kind of problem.

	3 feet 9 inches	
+	2 feet 7 inches	
	5 feet 16 inches	preliminary answer
	6 feet 4 inches	final answer

To reach the final solution, we needed to convert 16 inches to 1 foot 4 inches. The 1 foot is then "carried over" into the feet column. Similarly, if we made this into a base 12 problem, we would be thinking about grouping by 12, and carrying the number of groups over into the next column.

Example: Add $39_{12} + 27_{12}$.

Solution:

$$\begin{array}{r} 1\phantom{9_{12}} \\ 39_{12} \\ +\ \underline{27}_{12} \\ 64_{12} \end{array}$$

Writing numbers in a different base is similar to expressing ideas in a different language. The number (idea) doesn't change, only its symbolic representation. This analogy may also be helpful to consider as we think about doing arithmetic in different bases. Adding two numbers in base 7 is like answering a question posed in a new language. When I was first learning German, and the teacher asked a question in German to which I was to respond in German, I did something like this. I translated the question in my mind into English, thought about it and formulated the answer in English. Then I translated the answer into German, and responded. Once I was more experienced, I learned that I could think in German.

Here's the analogy. Given an addition problem in base 7, it would be possible to "translate" the numerals into base 10, do the addition as usual, and then "translate" the answer back into base 7.

Example:

$5_7 + 6_7$	$= 5_{10} + 6_{10}$	translation to base 10
	$= 11_{10}$	base 10 addition
	$= 14_7$	translation back to base 7

But the more sophisticated approach would be to learn how to add in base 7 directly. You could "think in base 7" this way: 5 plus 2 more is 1 group of 7, and you would have 4 left over; then you would simply write $5_7 + 6_7 = 14_7$. The "1 group of 7" is what is sometimes described in the addition algorithm as "carry the 1".

Example: $4_5 + 3_5 = 12_5$

Here's another example, mimicking an example from the more familiar base 10 presented first for comparison.

Example: Add 58 and 79 (base 10).

Solution:

$$\begin{array}{r} {}^{1\,1} \\ 58 \\ +\ \underline{79} \\ 137 \end{array}$$

Here 8 + 9 = one group of ten plus seven left over = 17, and the “1” is carried into the tens column. Now here’s, a similar problem involving the very same processes in base 7 (but carrying groups of sevens instead of tens).

Example: Add $65_7 + 56_7$.

Solution:

$$\begin{array}{r} {}^{1\,1} \\ 65_7 \\ +\ \underline{56}_7 \\ 154_7 \end{array}$$

Here 5 + 6 = one group of seven plus four left over = 14_7, and the “1” is carried into the sevens column.

Multiplication in different bases.

What about multiplication? The algorithm or procedure is the same as in base 10 (adjusted only to deal with groups of 7's instead of 10's). However, the basic "facts" are "different". The times tables you memorized long ago (unless you learned multiplication by something like the Egyptian system) were base 10. What we now need are base 7 times tables! We won't do this long enough to make memorization of times tables necessary or even efficient; but for reference, here they are for base 7 (I'll skip writing the subscripts just for now because everything in this table is base 7).

1•1=1	2•1=2	3•1=3	4•1=4	5•1=5	6•1=6
1•2=2	2•2=4	3•2=6	4•2=11	5•2=13	6•2=15
1•3=3	2•3=6	3•3=12	4•3=15	5•3=21	6•3=24
1•4=4	2•4=11	3•4=15	4•4=22	5•4=26	6•4=33
1•5=5	2•5=13	3•5=21	4•5=26	5•5=34	6•5=42
1•6=6	2•6=15	3•6=24	4•6=33	5•6=42	6•6=51

Now consider a simple multiplication problem; first an example from base 10 to recall the procedure, then a similar problem in base 7.

Example:

$$\begin{array}{r} {}^{22} \\ 34 \\ \underline{\times \ \ 7} \\ 238 \end{array}$$

Now in base seven, a similar problem involving the very same processes (but groups of sevens instead of tens):

Example:

$$\begin{array}{r} {}^{22} \\ 65_7 \\ \underline{\times \ \ 3}_7 \\ 261_7 \end{array}$$

Multiplication of two two-place numbers is just a bit longer.

Base 10:

$$\begin{array}{r} 45 \\ \underline{\times \ 72} \\ 90 \\ \underline{315\ \ } \\ 3240 \end{array}$$

Base 7:

$$\begin{array}{r} 65_7 \\ \underline{\times \ 32}_7 \\ 163_7 \\ \underline{261\ \ }_7 \\ 3103_7 \end{array}$$

“Decimals”

We will continue this discussion of different bases with some comments about writing "decimals". Remember, "decimal" is a base ten word, but we won't bother to make up a corresponding word in base 7. (We also won't do arithmetic with such numbers, although the procedures are essentially the same as with decimals.) First, some base 10 examples to remind ourselves what we mean:

$0.2 = \frac{2}{10}$ $\qquad$ $0.37 = 3 \bullet \frac{1}{10} + 7 \bullet \frac{1}{100} = 0.3 + 0.07 = 0.37$

In a similar fashion,

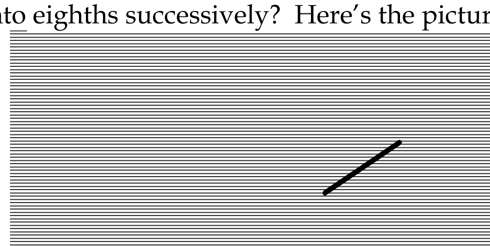
$0.2_7 = \frac{2}{7} = 0.2857143$

$0.35_7 = 3 \bullet \frac{1}{7} + 5 \bullet \frac{1}{49} = \frac{21}{49} + \frac{5}{49} = \frac{26}{49} = 0.5306122...$

$0.2_8 = \frac{2}{8} = 0.25$

$0.75_8 = 7 \bullet \frac{1}{8} + 5 \bullet \frac{1}{64} = \frac{56}{64} + \frac{5}{64} = \frac{61}{64} = 0.953125$

Converting from base 7 (or any other base) to base 10 is not that difficult. But what about going the other way? How, in general, does one convert a base 10 decimal into another base? For instance, how would you write one-third in base 8? Instead of grouping into groups of eight as we do on the whole-number side, in fractions we visualize cutting a pie into eight equal parts. So, how would you get to one-third of a pie by cutting into eighths successively? Here's the picture:



After the first cut into eighths, it is clear that one-third of the pie includes 2 of the "eighths", and a bit more. Now cut the piece containing the "bit more" into eighths. The one-third of the pie includes 5 of the "eighths of an eighth" (i.e., $\frac{1}{64}$th) of the pie, and a new "bit more".

The pie is getting too difficult to cut, so we need a mathematical process, an abstract algorithm, to accomplish the same thing. It looks like this.

$8 \times .333\ldots = 2.666\ldots = \textcircled{2} + .666\ldots$
1/8 's
$8 \times .666\ldots = 5.333\ldots = \textcircled{5} + .333\ldots$

The process would continue on, with the decimal part alternating between .666... and .333... . So the digits generated would be 2, 5, 2, 5, ... Hence, $\frac{1}{3}$ would be written in base 8 as $.2525..._8$.

To do this on your calculator, enter $\frac{1}{3}$ on your calculator by dividing 1 by 3. (Your calculator will probably compute this more accurately, i.e., to more decimal places, than if you simply enter .33333333.) Now multiply by 8. You should see 2.66666666 or 2.66666667. Record the 2, then subtract 2, the whole number part before the decimal point. Then multiply .66666666 by 8. You should see 5.33333333. Record the 5, subtract 5, and multiply by 8. Record the whole number part, subtract it, and multiply by 8. Keep going until you get a whole number when you multiply by 8 or notice a repeating pattern.

Example: Write 0.5 in base 7.

$7 \times .5 = 3.5 = 3 + .5$ 1/7's

$7 \times .5 = 3.5 = 3 + .5$ 1/49's

Since we would continue to generate 3's, we have $0.5 = 0.333..._7$.

Example: Write 0.32 in base 5.

$5 \times .32 = 1.60 = 1 + .60$

$5 \times .60 = 3.0 = 3$

So $.32 = .13_5$

Check: $0.13_5 = \frac{1}{5} + 3\left(\frac{1}{25}\right) = \frac{5+3}{25} = \frac{8}{25} = \frac{32}{100} = 0.32$

Example: Write $\frac{5}{18}$ in base 9.

Solution: $\frac{5}{18} = .2777.....$

$9 \times .277777 = 2.5 = 2 + .5$

$9 \times .5 = 4.5 = 4 + .5$

$9 \times .5 = 4.5 = 4 + .5$

So $\frac{5}{18} = 0.244....._9$

Example: Write $\frac{5}{18}$ in base 6.

Solution: $\frac{5}{18} = .2777....$

$6 \times .2777.... = 1.66.... = 1 + .66$

$6 \times .666.... = 4$

So $\frac{5}{18} = 0.14_6$

The above algorithm is analogous to our earlier process in which we divided by the base to count groupings. Here we multiply by the base to determine how many pieces of a certain size will be included.

Recall that we discovered earlier that a rational number can be written as either a terminating or repeating decimal. Now we observe that for a rational number, whether or not it is expressible as a terminating or repeating "decimal" depends on the base in which it is written. As a decimal, $\frac{2}{7}$ is repeating, .285714 285714 285714 285714....... But in base 7, it would be $.2_7$! On the other hand, $\frac{1}{2}$ is .5 as a decimal, but .33333333...... in base 7!

What base is it?

There is one more puzzle-type problem we would like to consider concerning different bases. Suppose you were an anthropologist studying a people group with whom you were not familiar. You do know that many cultures use a base other than 10 for their numeral system. You discover that they write numbers using the same symbols you are accustomed to using, but when you count 16 cows in a field, the owner writes 31 as the number. You are not so naive as to believe that the owner cannot count. Assuming there are no hidden complications (such as the more complicated system of the solar calendar of the Mayans in the next chapter), what base is used by these people ?

You could solve the problem by trial and error. Since the owner used the digit 3, the base must be at least 4. So try 4. $31_4 = 13$, so that's not it. $31_5 = 16$, so it would seem that these people used base 5. A more systematic approach would involve a little algebra, using "x" as the unknown base, and the value of the second column:

$$31_x = 16 \quad \rightarrow \quad 3 \bullet x + 1 = 16 \qquad 3x = 15 \qquad x = 5.$$

Why study different bases?

1. Different bases reflect cultural differences in real civilizations.
   Recall that the Babylonians used base 60, and that the Greeks adopted this system for the purposes of astronomy.
   The Hindus and Chinese used base 10.
   The Mayans used base 20 (see the next section).

2. They have applications in current technology.
   Computer scientists use base 2, base 8, and base 16.

3. Experiencing a new way of doing things helps you see the old way in a new light.
   a. The old way is not the only possible way.
   b. The old way is not necessarily the best way.
   c. The new way helps you understand the old way. (Many people learn a lot about the grammar of their native language when they study a second language.)

4. It helps you appreciate different ways of doing other things.

CHAPTER 10: Different Bases **Homework**

1. Convert the following base 7 numerals to base 10: a) 23_7 b) 40_7 c) 105_7 d) 236_7.

2. Convert the following base 8 numerals to base 10: a) 21_8 b) 204_8 c) 173_8.

3. Convert the following base 10 numerals to base 7: a) 25 b) 35 c) 48 d) 70.

4. Convert the following base 10 numerals to base 5: a) 25 b) 61 c) 200 d) 376

5. Perform the following additions in base 7:
   a) $15_7 + 56_7$ b) $44_7 + 33_7$ c) $26_7 + 136_7$.

6. Perform the following additions in base 5:
   a) $23_5 + 42_5$ b) $404_5 + 343_5$ c) $43_5 + 234_5$.

7. Perform the following multiplications in base 7:
   a) $5_7 \times 6_7$ b) $25_7 \times 36_7$ c) $16_7 \times 43_7$.

8. Perform the following multiplication in base 5:
   a) $4_5 \times 3_5$ b) $42_5 \times 33_5$ c) $21_5 \times 32_5$.

9. Perform the following multiplications in base 9:
   a) $5_9 \times 6_9$ b) $25_9 \times 36_9$ c) $16_9 \times 43_9$.

10. Perform the following multiplications in base 2:
    a) $11_2 \times 10_2$ b) $101_2 \times 100_2$ c) $111_2 \times 111_2$.

11. a) The numeral 34 is written in an unknown base and equals nineteen. Find the unknown base.
    b) The numeral 42 is written in an unknown base and equals thirty. Find the unknown base.
    c) The numeral 51 is written in an unknown base and equals forty-one. Find the unknown base.

12. a) The number 35 is written in an unknown base and equals forty-one. Find the unknown base.
    b) The numeral 68 is written in an unknown base and equals sixty-two. Find the unknown base.
    c) The number 63 is written in an unknown base and equals ninety-three. Find the unknown base.

13. Convert the following base 5 numerals into base 10:
    a) $.2_5$ b) $.01_5$ c) $.03_5$ d) $.42_5$

14. Convert the following base 4 numerals into base 10:
    a) $.11_4$ b) $.22_4$ c) $.03_4$ d) $.311_4$

15. Convert the following base 10 numerals to base 5:
    a) .2 b) .04 c) .24 d) .7 e) .024

16. Convert the following base 10 numerals to base 4:
    a) .5 b) .25 c) .3125 d) .3 e) .03125

Selected Answers:

1. a) 17	b) 28	c) 54	d) 125	
3. a) 34_7	b) 50_7	c) 66_7	d) 130_7	
5. a) 104_7	b) 110_7	c) 165_7		
7. a) 42_7	b) 1332_7	c) 1114_7		
9. a) 33_9	b) 1033_9	c) 720_9		
11. a) 5	b) 7	c) 8		
13. a) $\frac{2}{5} = .4$	b) $\frac{1}{25} = .04$	c) $\frac{3}{25} = .12$	d) $\frac{22}{25} = .88$	
15. a) $.1_5$	b) $.01_5$	c) $.11_5$	d) $.322..._5$	e) $.003_5$