

CHAPTER 21: Non-Euclidean geometry

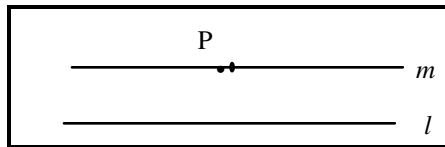
When I see the blindness and wretchedness of man, when I regard the whole silent universe, and man without light, left to himself, and as it were, lost in this corner of the universe, without knowing who has put him there, what he has come to do, what will become of him at death, and incapable of all knowledge, I become terrified, like a man who should be carried in his sleep to a dreadful deserted island, and should awake without knowing where he is, and without means of escape. And there upon I wonder how people in a condition so wretched do not fall into despair.¹⁰³

Morris Kline claims, "The two concepts which have most profoundly revolutionized our intellectual development since the 19th century are evolution and non-Euclidean geometry."¹⁰⁴ Everyone has heard of evolution; its impact on Christianity and society have certainly been profound and far-reaching. But what about non-Euclidean geometry? How many people even know there is such a thing as non-Euclidean geometry? Isn't geometry just geometry? What would non-Euclidean geometry look like? And even if there is such a thing, how important could a part of math be that hardly anyone has ever heard of??

For 2000 years, Euclidean geometry was regarded as certain truth. While Plato considered geometry to contain eternal truth about (non-physical) ideals, he saw that truth reflected in the world around us. Other Greek thinkers considered geometry to contain truth about the physical world. In either case, the truth of Euclid's geometry was not questioned. Descartes viewed Euclidean geometry as the paradigm for the way to obtain truth in any discipline. Its reputation could hardly have been better.

There was, however, what appeared to be a slight flaw in Euclidean geometry. The fifth postulate, which concerned parallel lines, seemed much more complicated than the other four geometric postulates. As stated earlier, Euclid assumed the following axiom:

AXIOM: Given a line l and a point P not on that line, there exists in the plane of P and l and through P one and only one line m , which does not meet the given line l .



Mathematicians starting with Euclid himself looked for a way to improve this situation. Some tried to prove the fifth postulate from the other four, which would have made it into a theorem. Others looked to replace it with something simpler and more obviously true. But no one questioned its truth.

But perhaps you are thinking that this statement IS obviously true. Let me try to convince you that it may not be as simple as you think.

First, what makes you believe two lines could ever really be parallel? Perhaps you are thinking of railroad tracks. One property I suspect you associate with parallel lines is that the (perpendicular) distance between them is always the same. If this weren't true for railroad tracks, the train would derail. So I am willing to grant that railroad tracks never meet. But railroad tracks aren't lines. Lines can always be extended. How do you know that the tracks aren't just a tiny bit closer after 1000 miles, so that if you extended them, say, another 1,000,000 miles, they WOULD meet?? Now that may be a ridiculous question for railroad tracks, but not for lines in geometry. What Euclid was asserting was that we should assume some lines, no matter how far we extend them, will NEVER MEET!! Is that really obviously true?? Does your experience with finite line segments like railroad tracks really provide you with enough information to

¹⁰³Pascal, Pensees, p. 236.

¹⁰⁴Kline, Mathematics for the Non-mathematician, p. 452.

assert what lines, which are (potentially) infinitely long, will be like?? Does it really make sense to believe that some lines will never, ever meet? Wouldn't it at least make some sense, maybe even just as much sense, to believe that any two lines would meet eventually?

Now consider for a moment the other possibility. If you believe there is a line m through the point P which is parallel to the line l , why would you believe there is only one such line? Why couldn't another line n also go through P and never meet l ? If you say that n must be getting closer and closer to l , I'll concede that for this discussion. So why does that mean that n and l must meet? I can almost hear you say it, "Well, eventually they must!" Why? Are you really sure about what happens in the realm of "eventually"? Maybe the lines do get closer and closer together, but never actually meet. If you say that I must be talking about a curve, that "straight" lines couldn't do that, then I would need to remind you that the axioms contain all that we know about lines. Whatever you think "straight" lines are like is probably a result of assuming Euclid's fifth postulate is true.

In any case, I hope you can see that the "truth" about parallel lines is a bit beyond our everyday experience. The assumption that lines can be extended indefinitely means that the issue relates to the concept of infinity. Euclid's geometry is not about the finite line segments of our everyday experience, but about lines which are potentially infinitely long.

Euclid showed his concern about the fifth postulate by first proving all the theorems he could without using it. Many other mathematicians who followed him tried to replace it or prove it. Perhaps the man with the most success in this effort was a Jesuit priest named Saccheri, who lived around 1700. Based on the need for consistency in mathematics, Saccheri tried the indirect approach to proving the fifth postulate. Assuming more than one parallel through the given point P , he deduced some very strange results. Unfortunately for his plan, none of these strange results was an outright contradiction of the other axioms or the theorems based on them.

Gauss's geometry

The first mathematician to draw the right conclusions about this situation was Carl Gauss (1777-1855). During his lifetime, Gauss was known as the "prince of mathematicians". His contributions to both pure and applied mathematics are astounding. Gauss realized at an early age that a postulate stating that there were TWO lines through the point P parallel to the line l was independent of the other axioms of geometry. That is, one could assume this postulate instead of Euclid's fifth postulate, and the resulting (non-Euclidean) geometry would be consistent, i.e., free of logical contradictions.

The rest of the story is an indication of how revolutionary this idea was, even to mathematicians. Gauss was the leading mathematical authority of his day. But he was afraid to publish his conclusions about a way to do geometry that contradicted Euclid. Even after other mathematicians made the same discoveries, he only acknowledged his work in private correspondence.

Gauss understood how the questions surrounding Euclid's parallel postulate related to infinity. He once wrote, "Finite man cannot claim to be able to regard the infinite as something to be grasped by means of ordinary methods of observation."¹⁰⁵ While Gauss believed in the mathematical legitimacy of his version of non-Euclidean geometry, he was never able to demonstrate that it was the geometry of the world in which we live.

Two other mathematicians, Nicolai Lobachevsky, a Russian, and Janos Bolyai, a Hungarian, independently developed the non-Euclidean geometry Gauss had discovered, and were the first to publicly claim the discovery. However, I will refer to this type of non-Euclidean geometry by Gauss's name for the sake of simplicity.

Some of the theorems of Euclidean geometry carried over into Gauss's non-Euclidean geometry, simply because the first four axioms were still the same. However, the substitution of Gauss's parallel postulate had far-reaching effects. Here are some of the contrasts:

Euclid: The sum of the angles of a triangle is 180° .

Gauss: The sum of the angles of a triangle is **less than** 180° .

¹⁰⁵cited in Eli Maor, *To Infinity and Beyond*, Birkhauser, Boston, 1987, p.131.

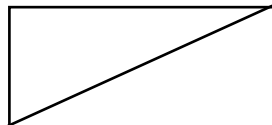
Euclid: Congruent triangles are similar, but similar triangles are not necessarily congruent.

Gauss: Congruent triangles are similar, and similar triangles are congruent. That is, the two concepts of similarity and congruence are indistinguishable.

The first of these contrasts, that the sum of the angles of a triangle is less than 180° , is the most significant for our purposes. Here's a rather strange and surprising implication: in Gauss's geometry, there are no rectangles! By a rectangle, I mean a four-sided figure with four right angles.

Theorem 1: Rectangles do not exist (in Gauss's geometry).

Proof: Suppose for the moment that there was a rectangle. Draw a diagonal (the line that joins opposite corners):



Now you have two triangles. The sum of the angles in each triangle is less than 180° . So the sum of the angles of the two triangles is less than 360° . However, the angles of the two triangles clearly equal the angles of the rectangle, and they add up to 360° ! Since the sum of the angles of the rectangle can't be both equal to and less than 360° , we have a contradiction. Hence the assumption that we could start with a rectangle must be mistaken. There is no such thing in Gauss's non-Euclidean geometry.

Well, this may be starting to sound a little too weird for some of you. You may actually think you've seen a rectangle! But are you sure? Remember, every angle would need to be exactly 90° . Not just close, not just 90° to the closest degree, but exactly. Counting can be an exact process (there are exactly 15 words in the first sentence of this paragraph), but measurement rarely (if ever) is.

Riemann's geometry

Nor are we done. Another mathematician, Bernard Riemann, discovered a second type of non-Euclidean geometry. Around 1850, Riemann realized that another logically consistent version of geometry could be based on the other assumption one could make about parallel lines, that they don't exist. Riemann found that in order to make this work, however, he had to make some other alterations in Euclid's axioms. In particular, Riemann removed the potentially infinite length of lines. In Riemann's geometry, lines have a finite length, and all lines intersect.

Here are some of the other contrasts which occur:

Euclid: The sum of the angles of a triangle is 180° .

Riemann: The sum of the angles of a triangle is **greater than** 180° .

Euclid: Congruent triangles are similar, but similar triangles are not necessarily congruent.

Riemann: Congruent triangles are similar, and similar triangles are congruent. That is, the two concepts of similarity and congruence are indistinguishable.

Euclid: If two lines are perpendicular to the same line, then they are parallel.

Riemann: If two lines are perpendicular to the same line, then they intersect. In fact, all lines perpendicular to a given line intersect at the same point.

By the way, there also are no rectangles in Riemann's geometry. Think about it this way. In Euclidean geometry, another way of talking about rectangles is to say that they have two pairs of parallel sides. But there are no parallel lines in Riemann's geometry. So how could there be rectangles?

Now I want you to use your imagination. Think back with me to Flatland, a world of two dimensions. You recall that the Sphere took A. Square out of Flatland, so that from Spaceland, the world of three dimensions, he was able to look down on his own house and his own world. Suppose the Sphere only took A. Square up a short distance, maybe 50 feet (A. Square wasn't much over a foot square). Of course from that height, Flatland would look flat, much like our earth would look flat from that height. So Flatland would look flat even if it was a two-dimensional surface stretched over a very large sphere. What I'm suggesting is that Flatland might conceivably be a place like our earth, except that there was no height. Now here's the real question: what kind of geometry works in such a place?

Remember, Flatland is two-dimensional, so we are talking about plane geometry, just like we have been discussing. So, imagine for the moment that A. Square lives on the "equator". (I'm using earth language, of course; A. Square thinks of his world as flat.) Suppose he takes a trip, going due east. After a while, he makes a 90° left turn. He continues traveling straight to the "North Pole". Checking his map, he notices that if he now turns and heads straight for home, his path will be a line perpendicular to "the equator". This trip, traced on his map, is a triangle. This triangle has three angles, two of which are right angles. Clearly the sum of the angles of this triangle would be greater than 180° . If this is what Flatland is really like, it's geometry would be Riemann's!!

Now back to our world and reality. Perhaps you are thinking that even if these non-Euclidean geometries are logically valid, they are mere mathematical curiosities, and have had little impact outside of mathematics. Not true. Recall that we began this section with a quote from Morris Kline claiming that non-Euclidean geometry is one of the two most significant revolutionary concepts of the last 200 years. It's time to discover how he could make such a claim.

Geometry and the "real world"

First, let's address the question of the geometry of the world in which we live. For two thousand years after it was formulated, essentially everyone believed that Euclidean geometry was a true description of our world. Of course, they had no viable alternative to consider. However, even after the discovery of non-Euclidean geometries, most people of the 19th century still assumed Euclid was true, and these other geometries were nice, but not true of our world. Gauss tried to verify that non-Euclidean geometry was the geometry of our world. He failed to get decisive results. However, ever since some observations of certain stars in 1919, that has all changed. Those star observations provided confirming evidence for Albert Einstein's theory of relativity, a theory about how our world operates. Using the path of light as the physical interpretation of "line", Einstein developed a theory whose geometry was not Euclidean. It was, rather, a version of Riemann's geometry. And in 1919, it became apparent that the geometry of our world is most likely Riemann's, not Euclid's.

Now the reason that it took star observations to discover this is that the differences in the three geometries are likely to be undetectable over short distances. It's only at the astronomical level that there is enough of a difference to notice. So if you want to keep using Euclidean geometry in your daily life, that will probably be just fine. But if you ever have a chance to take a trip to the moon, you might just want to check with the captain to make sure HE believes in non-Euclidean geometry!

If the influence of non-Euclidean geometry extended only to the physics of the universe, it probably wouldn't be that big a deal to many people. However, that's only the beginning.

For mathematics, the discovery of logically consistent alternatives to Euclid was freeing. Mathematics was no longer tied to physical reality. At most one of the options could be the real geometry of space, but from a mathematical point of view, all three were legitimate. The criteria by which mathematics was judged were freed from the attempt to study the real world; other criteria, based on

aesthetical considerations, were allowed. Georg Cantor, a mathematician whom we will discuss at greater length in Chapter 25, went so far as to assert that, "The essence of mathematics is its freedom."¹⁰⁶

On the other hand, it now became apparent that the statements of mathematics were not automatically "true". For many, many years, philosophers (and many other people as well) had taken the statement " $2 + 2 = 4$ " as an example of a statement which is obviously true. Hence any philosophical discussion about truth had to reckon with " $2 + 2 = 4$ ". But people had been equally certain about "The sum of the angles of a triangle is 180". If Euclidean geometry was not "true", Descartes' plan to have a method to obtain certain truth in philosophy and other areas was shaken, to say the least. The question entered people's minds, "If we can't decide which set of fundamental geometric axioms to use (and hence which geometry is true), then how can we decide among conflicting metaphysical or religious "axioms"? If there isn't any truth to be found in mathematics, what chance is there of finding truth in religion, or ethics, or any of a number of other fields?

At a time when European civilization was no longer solidly committed to a Christian perspective, the discovery of equally logical alternatives to Euclidean geometry suggested to some people that maybe there were no absolutes after all. Maybe there wasn't even any truth.

Let me hasten to say that I do not believe all that has been suggested in the last two paragraphs. I do believe in truth, I do believe in absolutes. I do believe (in particular) that Christianity is true, and that "salvation is found [in Jesus Christ and] in no one else, for there is no other name under heaven given among men by which we must be saved."¹⁰⁷

To reason from the existence of non-Euclidean geometries to the non-existence of truth is an unjustified leap. On the other hand, Descartes had thought too highly of Euclidean geometry. The way to certain truth is not through a method which mimics Euclid. Euclidean geometry was an amazing achievement, but it was only mathematics, not the answer to all the aspirations of the human race. Absolute certainty, if we can have it, must come from somewhere else.

There is something quite humbling about this whole story. For over 2000 years, many people believed Euclidean geometry was the one and only true geometry. Many of these people were quite intelligent, people we respect. But they were WRONG. Have you ever wondered what basic truths you know (that is, think you know) that might turn out to be wrong? That thought makes me just a little less sure of myself and my opinions. And I think that's a good thing.

On the other hand, there is something quite impressive about the discovery of non-Euclidean geometries. The creators of non-Euclidean geometries pursued the results of reason in spite of tradition, philosophical doctrines, and "common sense". Gauss and others were able to overcome these obstacles and see things from a completely different perspective. Reason is a wonderful gift from God. We should use it wisely.

(No Homework for this chapter.)

¹⁰⁶quoted in Joseph Dauben, Georg Cantor, Princeton University Press, Princeton, 1990; p. 161.

¹⁰⁷Acts 4:12; parenthetical insertion is mine, for clarification.