Therefore I shall not undertake here to prove by natural reasons either the existence of God, or the Trinity, or the immortality of the soul, or anything of that nature; not only because I should not feel myself sufficiently able to find in nature arguments to convince hardened atheists, but also because such knowledge without Jesus Christ is useless and barren. Though a man should be convinced that numerical proportions are immaterial truths, eternal and dependent on a first truth, in which they subsist, and which is called God, I should not think him far advanced towards his own salvation.\textsuperscript{35}

Algebraic notation began to appear in western Europe around 1500 A.D. Before then, algebra was written in words, with some abbreviations being commonly accepted. How would you like to work with "equations" like this: when the unknown is multiplied by three, and then six is subtracted, the result is the same as if the unknown were multiplied by two, and then subtracted from nine. $3x - 6 = 9 - 2x$ should seem like a vast improvement. By the way, do you know the answer? "Three" is correct. Did you work with the "words" or "symbols"?

There are several reasons why algebraic notation only gradually replaced words. For us, it is obviously simpler and easier to understand-- once you learn it. The same was true historically. Change does not come easily for most of us, and so people who grew up solving algebra problems in words were not very inclined to adopt the new abbreviations or notation.

The case can be made that algebraic notation is actually more clear and unambiguous. As one example with numbers, what’s three times four plus five? Did you say 17 or 27? I meant 27, because I wanted you to do $3 \cdot (4+5)$. But how do you distinguish this from $(3 \cdot 4) + 5$ in words? Algebraic notation makes it easy.

Finally, algebraic notation is an extremely powerful tool. Good notation can actually suggest ways to solve problems. Exponent notation, the subject of the previous chapter, is an example of this. Before we were done, we saw how it helped us understand how radicals work. We will experience this again in this chapter.

Algebra is arithmetic done with unknown numbers. This statement has one very practical implication. The rules for algebra are not arbitrary. They follow the rules for arithmetic, which are derived from certain types of everyday experience. For instance, $(2+3) \cdot (4+5) = 45$. If all the numbers in an expression are known, certain steps to simplify are possible which would not be possible if some numbers were unknown. So instead of adding inside the parentheses to get $5 \cdot 9 = 45$, consider the following approach to the same problem. It is more complicated, but still works:

$$(2+3) \cdot (4+5) = 2 \cdot 4 + 2 \cdot 5 + 3 \cdot 4 + 3 \cdot 5 = 8 + 10 + 12 + 15 = 45.$$

This is the approach we need if some numbers are unknown, like this:

$$(x+3) \cdot (x+5) = x \cdot x + 5x + 3x + 3 \cdot 5 = x^2 + 8x + 15.$$

One common way to remember this process is “FOIL”:

- **F** multiply the *FIRST* terms in each expression
- **O** multiply the *OUTSIDE* terms
- **I** multiply the *INSIDE* terms
- **L** multiply the *LAST* terms

\textsuperscript{35}Pascal, \textit{Pensees}, p. 184-185.
Algebra can act as a guide in our thinking. An interesting, but not very important, illustration of this is the following “number trick”. You may remember number tricks from elementary school when you were told to think of a number and this was followed by a series of operations to perform and, in the end, everyone ended up with the same number provided no one made a mistake. Here’s a relatively simple example.

1. Pick a number.
2. Multiply it by 2.
3. Add 4.
4. Add the original number.
5. Add 5.
7. Subtract the original number.
8. You ended with the number 3.

Algebra explains why we all ended up with the same number.

- Pick a number. Since we don’t know the number selected, call it \( x \)
- Multiply by 2. \( 2x \)
- Add 4. \( 2x + 4 \)
- Add the original number. \( 3x + 4 \)
- Add 5 \( 3x + 9 \)
- Divide by 3 \( x + 3 \)
- Subtract the original number. \( 3 \)

Could we have constructed this rather simple problem without algebra and be confident of its outcome? Remember, we were told to pick a number, any number; positive, negative, zero, rational, irrational. Could I have started with a particular number, made up the steps, and know for sure that I would always end up with 3 no matter what number I started with? Algebra guarantees it!

The following is a more elaborate and interesting example:

1. Start with the number of your month of birth.
5. Subtract 200.
6. Add the number of the day of your birth.
7. Multiply by 2.
8. Subtract 40.
9. Multiply by 50.
10. Add the last two digits of the year of your birth.
11. Subtract 10500.

Look at your ending number and you should see the month, day and year of your birth provided, of course,
that you haven’t made a mistake. We are confident that we didn’t make a mistake. Why are we so confident? Let’s look at what’s going on.

Start with the number of the month of your birth. \( x \)
Multiply that number by 4 \( 4x \)
Add 13 to the result. \( 4x + 13 \)
Multiply the result by 25 \( 100x + 325 \)
Subtract 200 \( 100x + 125 \)
Add the number of the day of your birth. \( 100x + y + 125 \)
(Since we don’t know what it is, we’ll add \( y \).)
Multiply by 2. \( 200x + 2y + 250 \)
Subtract 40. \( 200x + 2y + 210 \)
Multiply by 50 \( 10000x + 100y + 10500 \)
Add the last two digits of the year of your birth. \( 10000x + 100y + z + 10500 \)
(We don’t know it either, so we’ll use \( z \).)
Subtract 10500. \( 10000x + 100y + z \)

If you were born on June 16, 1981, for example, then your value for \( x \) is 6, \( y = 16 \) and \( z = 81 \). Therefore we have \( 6(10000) + 16(100) + 81 = 60000 + 1600 + 81 = 61681 \). Kind of nice. We could not have done it without algebra!

Solving equations

A basic rule about working with equations is to do the same operation to both sides. This is entirely reasonable, since in an equation you are asserting two things are equal, and therefore to keep them equal, you would need to treat them equally.

A basic strategy to solve an algebra equation involves the following steps:

1) remove parentheses appropriately
2) remove fractions by multiplying both sides of the equation by the same (carefully chosen) number
3) get all the \( x \) terms on one side of the equation and all the numbers on the other side
4) solve for one \( x \) by dividing both sides of the equations by the coefficient of \( x \)

Example 1: Solve \( 3x = 12 \).

Solution: Divide both sides by 3 to find out what one \( x \) would equal:

\[
\frac{3x}{3} = \frac{12}{3} \quad \Rightarrow \quad x = 4
\]

Check: \( 3(4) = 12 \).

Example 2: Solve \( 2(x + 3) = 18 \).
Solution: First, remove the parentheses by multiplication:

$$2x + 6 = 18.$$ 

Now divide both sides of the equation by 2:

$$x = \frac{12}{2} = 6.$$ 

Check: $2(6 + 3) = 2(9) = 18.$

Example 3: Solve $\frac{5}{x-2} = 7.$

Solution: First, remove the fractional appearance of the problem by multiplying both sides of the equation by $x - 2$:

$$(x - 2) \cdot \frac{5}{x - 2} = 7 \cdot (x - 2)$$

Multiply out on both sides:

$$5 = 7x - 14$$

Add 14 to both sides:

$$19 = 7x$$

Divide both sides by 7:

$$\frac{19}{7} = x.$$ 

Solving Word Problems

Solving word problems successfully involves both verbal and analytical skills. Many students approach word problems as if the fact that this is a math class means that you must start solving a problem by immediately writing down numbers, and letters that represent numbers. For many people, there is a better way. Here’s an outline of an approach I think you might find helpful. Give it a try if word problems make you nervous.

1. **Read and understand** the problem.
   - Is the situation in the problem one with which you are familiar? If not, read it again.
   - Do you understand how the various parts of the problem are related? For instance, if the problem is about gasoline at $1.45$ per gallon, do you know that you would multiply that price times the number of gallons you purchase to determine how much you will need to pay?
   - Do you know what you are asked to find? If you aren't perfectly clear on your objective (the unknown in the problem), your chances of finding go way down.
   - Do you have a rough estimate of what the answer will look like (about how big it will be)? For instance, if gasoline is $1.45$ per gallon, and you have $5$, it would be helpful if you knew you would be able to buy a little bit more than 3 gallons of gasoline.

2. **Organize the parts** of the problem, and if possible, draw a "picture".
Since many of us "see" things better in a picture, try drawing even a rough sketch of the situation, or a displaying the parts of the problem in a chart or diagram. Very often, a well-drawn picture will make the next step much easier.

Identify the unknown with a letter (often x is used); make sure you include it as a part of the problem in your picture.

3. **Write an equation** involving the unknown.
   
   An equation is a statement that two quantities are equal. In many problems you will be able to discover two ways of computing the same quantity; your equation will be a statement that these two are equal. In other problems you may need to use a special formula. In any case it is often helpful to write out the equation as a sentence using words before trying to write it as an equation using algebraic notation.

4. **Solve the equation** for the unknown.
   
   Let the notation and rules of algebra be your guide here.

5. **Check your solution** to make sure your answer is the answer to the question the problem asked.
   
   First, check to make sure you did the algebra correctly. Substitute your answer into your original equation to make sure it works, or double-check your work.
   
   Then see if your answer fits the estimate you made of the answer back in step #1. If the two numbers are not close, consider whether you might have erred significantly in your estimation. Or, think again about the equation you wrote down.
   
   Once you have resolved the difference in your estimate and your answer, or if they were close, then check to see whether the answer works in the original word problem. Sometimes this would be redundant, but sometimes it may not look that way, so it is a final way of convincing yourself that you have indeed succeeded in solving the problem!

**Example 4:** Suppose a student has test scores of 80 and 86. How low a grade can this student get on a third test and still have a 75 average?

**Solution:** Let \( x \) = score on the third test.

\[
\frac{80 + 86 + x}{3} = 75
\]

Multiply each side of the equation by 3:

\[
166 + x = 225 \quad x = 225 - 166 = 59.
\]

**Check:** \( 80 + 86 + 59 = 225 \) \( \frac{225}{3} = 75 \).

**Example 5:** In a certain course, there are two exams worth 60% of the semester grade, a research paper worth 25% and short quizzes worth 15%. Suppose a student got an 88 on the first test, a 94 on the research paper and had a 78 average on the quizzes. How low a score can the student get on the second test to have a 90 average for the semester.

**Solution:** Let \( x \) = score on second test.

An equation is set up by multiplying each score by the percentage it contributes to the semester average, and setting the sum equal to the desired average.
0.30 (88) + 0.30x + 0.25 (94) + 0.15(78) = 90
26.4 + 0.3x + 23.5 + 11.7 = 90
0.3x = 90 - 61.6
0.3x = 28.4
x = \frac{28.4}{0.3} = 94.66
So if the professor doesn’t round off, the student needs a 95.

Example 6: John just got a 5% raise at work. Now he makes $8.82/hr. At what rate was he being paid before the raise?

Solution: Let x = John's original hourly rate.

Original hourly rate = x  
raise = 0.05x

New hourly rate = $8.82

original hourly rate plus hourly raise equals new hourly rate
x + 0.05x = 8.82
1.05 x = 8.82
x = \frac{8.82}{1.05} = 8.40, that is, $8.40/hr.

Example 7 (review): In an unknown base, 25 (base 10) is written as 31. What is the unknown base?

Solution: Let x = unknown base

31_x = 25  
3x + 1 = 25  
x = 8

Example 8: A person has 20 coins in a cup worth a total of $1.60. The coins are a mixture of dimes and nickels. How many of each are in the cup?

Solution: Let x = the number of dimes in the cup. Then
20 - x = the number of nickels in the cup.

x dimes   (20 - x) nickels
worth .10 x  worth .05 (20 - x)

The value of the dimes plus the value of the nickels equals the total value of the coins.
(0.10 \cdot x) + (0.05 \cdot [20 - x]) = 1.60.

0.1x + 1 - 0.05x = 1.60
0.05x = 0.60
x = \frac{0.60}{0.05} = 12 \text{ dimes}

20 - x = 20 - 12 = 8 \text{ nickels.}

Check: 12 dimes is $1.20, and 8 nickels is $0.40, for a total of 20 coins worth $1.60.

**Example 9:** A car's radiator currently contains 3 quarts of liquid which is 35% antifreeze. How many quarts of antifreeze must be added to increase the antifreeze content to 55%?

**Solution:** Let \( x \) = the amount of antifreeze to be added.

3 + x = the amount of liquid in the radiator after adding antifreeze

\[
\begin{array}{|c|c|}
\hline
\text{100\%} & \text{x} \\
\hline
\text{35\%} & 3 \text{ qt} \\
\hline
\end{array}
\} 55\% (3+x)
\]

The original amount of antifreeze plus the added antifreeze equals total antifreeze.

\[
(0.35)(3) + x = (0.55)(3 + x)
\]

1.05 + x = 1.65 + 0.55x

0.45x = 0.60

x = \frac{0.60}{0.45} = 1.33 \text{ quarts}

**Example 10:** Edith is mixing a punch which contains three times as much orange juice as pineapple juice. If she wants 10 liters of the mixture, how much pineapple juice should she use?

**Solution:** Let \( x \) = the amount of pineapple juice to be used.

3x = the amount of orange juice to be used

\[
\begin{array}{|c|c|}
\hline
\text{pineapple juice} & x \\
\hline
\text{Orange juice} & 3x \\
\hline
\end{array}
\]

= 10 liters
orange juice + pineapple juice = total mixture
\[ 3x + x = 10 \]
\[ 4x = 10 \]
\[ x = 2.5 \text{ liters of pineapple juice.} \]

CHAPTER 14: Algebra

Homework

1. Solve the equations for x.
   a. \[ 5x + 3 = 13 \]
   b. \[ 3x - 5 = 10 \]
   c. \[ 2(x+3) = 14 \]
   d. \[ 3(x - 2) = 10 \]

2. Solve the equations for x.
   a. \[ 5 - 2x = 17 \]
   b. \[ 3(x + 2) = 9 \]
   c. \[ 2(x - 3) = 8 \]
   d. \[ 10x - 8 = 12 \]

3. Solve the equations for x.
   a. \[ .25(76) + .75x = 82 \]
   b. \[ \frac{1}{2}x - \frac{1}{3} = \frac{1}{4} \]
   c. \[ 0.05(13 - x) = .2 \]
   d. \[ \frac{2}{3}x + \frac{1}{2} = 1 \]
   e. \[ 0.25x + 0.1(20 - x) = 2.75 \]

4. Solve the equations for x.
   a. \[ 0.20(x + 2) + 0.05(10 - x) = 1.4 \]
   b. \[ 0.5(14 - x) - 0.3(x + 4) = 17.8 \]
   c. \[ \frac{1}{4}x - \frac{2}{3} = \frac{1}{6} \]
   d. \[ \frac{2}{5}(x - \frac{1}{2}) = 13 \]

5. Solve the equations for x.
   a. \[ \frac{2}{x} = 10 \]
   b. \[ \frac{4}{x} = \frac{1}{8} \]
   c. \[ \frac{12}{x+2} = 2 \]

6. Solve the equations for x.
   a. \[ \frac{10}{x - 3} = 4 \]
   b. \[ \frac{3}{x} = \frac{7}{2} \]
   c. \[ \frac{5}{x} = \frac{1}{3} \]

7. Perform the following multiplications.
   a. \[ (x + 4)(x + 2) \]
   b. \[ (x - 2)(x + 3) \]
   c. \[ (x - 1)(x - 4) \]
   d. \[ (2x - 1)(x + 3) \]

8. Perform the following multiplications.
   a. \[ (x - 2)(x - 3) \]
   b. \[ (3x - 2)(x + 1) \]
   c. \[ (x + 3)(x + 5) \]
   d. \[ (x - 3)(x + 2) \]

9. A student has grades of 70 and 83 on two tests. What grade must she get on a third test to have an average of 80?

10. A student has grades of 81, 89, and 77 on three tests. What grade must she get on the fourth test to have an average of 85?

11. Sales tax is 6%. If you purchase an item and the total charge (price plus tax) is $47.70, what was the price of the item?
12. Sales tax is 8%. If you purchase an item and the total charge (price plus tax) is $11.61, how much tax did you pay?

13. When Harry goes bowling, he rents shoes for $2.00 and pays $2.50 per game. How many games can Harry bowl if he has $17?

14. First class postage is 41¢ for the first ounce and 17¢ for each additional ounce (or part of an ounce). How much can a first class item weigh if you can spend $2.13 to mail it? (Your answer should be a whole number of ounces.)

15. Mary is mixing a punch which contains twice as much fruit juice as soda. If she wants 12 liters of punch, how much soda does she need?

16. Terry is mixing a punch which contains 3 liters of juice for every 2 liters of soda. If he wants to make 21 liters of punch, how much juice does he need?

17. A car’s radiator currently contains 4 quarts of liquid which is 40% antifreeze. How many quarts of 100% antifreeze must be added to increase the antifreeze content to 50%?

18. A car’s radiator currently contains 2 quarts of liquid which is 25% antifreeze. How many quarts of 100% antifreeze must be added to increase the antifreeze content to 60%?

19. Juan has grades of 95, 90 and 91 on three tests. What grade must he get on a fourth test to have an average of 93?

20. Sam had grades of 70, 81, and 77 on three tests. The final exam is worth as much as two tests. What grade must Sam get on the final to have an average of 85?

21. Suppose in a class you have grades of 83 and 77 on the tests, a 93 on the paper, and a homework grade of 71. [Assume: The three tests are worth 70%, the homework 20% and the paper 10%] What grade must you get on the third test to have an 82 average?

22. Suppose in a class you have test grades of 65 and 73, an 85 on the paper, and a homework grade of 40. [Assume: The three tests are worth 70%, the homework 20% and the paper 10%]. What grade must you get on the third test to get a semester average of 70?

23. Sam has 28 coins. Some are nickels and the rest are dimes. If the coins are worth $2.05, how many nickels does Sam have?

24. Amy has 38 coins. Some are nickels and the rest are quarters. If the coins are worth $5.70, how many are quarters?

25. Sue has 33 coins. Some are nickels and the rest are quarters. If the coins are worth $3.85, how many quarters does Sue have?

26. Jaime has 29 coins. Some are dimes and some are quarters. If the coins are worth $5.30, how many are dimes?

27. John has 21 coins. Some are quarters and some are dimes. All together they are worth $4.20. How many of each are there?
28. Sarah has 26 coins. Some are pennies and some are nickels. If the coins are worth 62¢, how many are nickels?

29. In a certain course, there are 3 exams worth 30% each and an oral presentation worth 10%. If a student got scores of 95 and 88 on the first two exams, and a 92 on the oral presentation, what is the lowest score she could get on the third exam and still have a 90 average?

30. In a certain course, there are two exams worth 30% each, a major paper worth 25% and quizzes worth 15%. If a student got scores of 84 on the first exam, and a 78 on the paper, and an average of 72 on the quizzes, what is the lowest score the student can get on the second test to have a semester average of 82?

Selected Answers:

1. a. 2 b. 5 c. 4 d. $\frac{16}{3}$

3. a. 84 b. $\frac{7}{6}$ c. 9 d. $\frac{3}{4}$ e. 5

5. a. .2 b. 32 c. 4

7. a. $x^2 + 6x + 8$ b. $x^2 + x - 6$ c. $x^2 - 5x + 4$ d. $2x^2 + 5x - 3$

9. 87 11. $\$45.00$ 13. 6

15. 4 17. $.8 = \frac{4}{5}$ 19. 96 21. 91 23. 15 25. 11

27. 14 quarters, 7 dimes 29. 87