CHAPTER 15: Euclidean Geometry

There are then a great number of truths, both of faith and of morality, which seem contradictory, and which all hold good together in a wonderful system. The source of all heresies is the exclusion of some of these truths;...

The word "geometry" suggests the original meaning of the term: "geo" refers to the earth and "metry" refers to measuring. In the ordinary experiences of life people had occasion to measure parts of the earth, like roads, fields, and water. Length, area, and volume were abstract concepts needed to measure one-, two-, and three-dimensional objects. Geometry was born of these experiences and contemplation. Formulas (or more properly in the historical context, procedures) were approximate in many cases. On the other hand, the engineering accomplishments of ancient cultures suggest that their approximations were apparently quite adequate in many cases.

It is to the classical Greeks to whom we owe geometry as a science or systematic body of knowledge. Earlier civilizations did not seem to separate geometrical problems from other mathematical information, nor organize them to see their inter-relatedness. Unlike the Babylonians and Hindus who favored working with numbers, the Greeks clearly preferred their systematic approach to geometry over number theory and earlier geometric work. Several reasons suggest themselves: first, geometry as the Greeks developed it dealt with abstract concepts, and abstract concepts can be treated with exactness. Both abstract and exact thinking appealed to the Greek mind. Second, after the relatively short time that the Pythagorean theme that "everything is number" was a major influence, geometry seemed to the Greeks to hold the key to the nature of the cosmos. Finally, since the Greeks never found an effective way to deal with irrational numbers arithmetically, they subsumed arithmetic and algebra under geometry. This approach which made geometry king in mathematics held sway in western Europe well into the Renaissance. (Even Isaac Newton, who around 1700 AD discovered the universal law of gravitation by his newly developed calculus, reverted to geometry when he was publishing his results.)

Euclidean Geometry

Euclidean geometry is named after Euclid (c. 300 BC), not because he discovered it all, but because he completely and definitively organized what was already known and filled in whatever gaps had existed. Thales, the father of mathematics, had proven some key theorems three centuries earlier. The Pythagoreans had added contributions, as did other lesser known mathematicians.

Euclid began with some naively simple definitions. He defined "point", for instance, as that which has no parts. While this may convey something of the concept of "point", it is inadequate as a precise definition. In fact, the modern perspective is that one must start with certain "undefined" terms; "point" would be one such term in geometry.

Then Euclid listed 10 axioms. Five of them were geometric in nature; five of them were more general in the sense that they applied to everything, not just geometrical things. Recall that axioms were viewed as self-evident truths, statements that any rational person would accept.

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36Pascal, Pensees, p. 304.
EUCLID’S 5 MATH AXIOMS

1. Two points determine a unique straight line.
2. A straight line can be extended as far as necessary in either direction.
3. A circle may be drawn with any given center and any given radius.
4. All right angles are equal.
5. Given a line \( l \) and a point \( P \) not on that line, there exists in the plane of \( P \) and \( l \) and through \( P \) one and only one line \( m \), which does not meet the given line \( l \).\(^{37}\)

EUCLID’S 5 GENERAL AXIOMS

6. Things equal to the same thing are equal to each other.
7. If equals are added to equals, the sums are equal.
8. If equals are subtracted from equals, the remainders are equal.
9. Figures which can be made to coincide are congruent.
10. The whole is greater than any part.

Look again at Axiom 2. There is a concept called potential infinity which fits Euclid’s lines. At any moment in time, a Euclidean line was finite in length, but with the potential for being extended, if necessary, to a greater (finite) length. This means that at no moment in time is the line ever really infinite in length. This may seem like a fine distinction, but it was a necessary one (from the ancient Greek point of view) to keep the actual infinite out of mathematics. On the other hand, in many modern textbooks, lines are presented as actually infinitely long. Philosophically, this is an important distinction. We shall continue this discussion of potential vs. actual infinity later. And we shall see that Axiom 10 will also play a crucial role in that discussion.

Recall the proof which we did earlier (see Chapter 4) that the sum of the interior angles of a triangle is 180°. The first step in the proof utilized Axiom 5 above in constructing a line through the point A parallel to the base of the triangle (see the diagram below).

\[\text{Diagram: } A \quad \text{line} \quad B \quad \text{triangle} \quad C\]

\(^{37}\)This version of Axiom 5 is actually due to the mathematician John Playfair, who lived around 1800 AD. It is logically equivalent to Euclid’s statement, more commonly known, and easier to use for our discussion.
Axiom 5 will be discussed in greater detail later when we consider what is called “non-Euclidean geometry”. For now, it is sufficient to notice that it is the longest and most complicated of the axioms. For this reason, the Greeks and mathematicians after them hoped to either prove it from the other axioms or replace it with a shorter, simpler axiom that would work as well. Neither attempt succeeded. Later I will go so far as to suggest that it is not nearly as self-evident as the other axioms. It will also be important to remember for that later discussion that Axiom 5 was used to prove that the sum of the angles of a triangle is 180°.

**Congruence**

Two geometrical figures are **congruent** if they have the same size and shape. Much of Euclid's efforts were spent on deriving conditions under which two triangles are congruent. If they are congruent, then corresponding angles measure the same, and corresponding sides have the same length. As Axiom 9 expresses it, figures which are congruent can be made to coincide. That is, if you think of moving one figure “on top of” the other, they would match exactly. The conditions Euclid sought enabled him to determine if two figures were congruent when they couldn’t actually be moved.

**Similarity**

Two geometrical figures are **similar** if they have the same shape (but not necessarily the same size). By "same shape", I don't mean merely something like "rectangle" or "triangle". It is not true that every two rectangles are similar. More officially, two geometric figures are similar if their corresponding angles measure the same, and if their corresponding sides are proportional in length. When a document is enlarged or reduced in size by a photocopying machine, what is produced is a similar figure. Within the category of "rectangle" there is more variation than the geometric concept of "similar" allows. As it turns out, the conditions under which two triangles are similar are fairly simple. Two triangles are similar if their corresponding angles measure the same. (Notice this criteria doesn't work for rectangles, since every rectangle has 4 right angles.)

Here are some examples:

**not similar:**

```
\[ \triangle  \quad  \triangle \]
```

**similar:**

```
\[  \triangle  \quad  \triangle \]
```

```
\[ \text{not similar:} \quad \text{similar:} \]
```

```
not similar: 

You've probably enjoyed watching "similar" shapes many times: when you watch a movie on a large screen, what you are looking at is "similar" to the images on the film running through the projector. The shapes are the same; but the images on the screen are simply much larger.

What is the relationship between the concepts “congruent” and “similar”? The answer for Euclidean geometry should make sense in the light of our discussion. Two congruent figures are similar, but two similar figures are not necessarily congruent.

Similar triangles have ratios of lengths of sides that are equal. For example,

\[
\begin{array}{c}
5 \\
2 \\
\end{array}
\quad \frac{2}{5} \\
10 \\
4 \\
\end{array}
\]

are similar triangles. The ratio of the lengths of the sides in the first triangles is \(\frac{2}{5}\). The ratio of the lengths of the corresponding sides is equal to \(\frac{4}{10}\) in the second triangle. This property helps us solve a certain type of problem. Sometimes there is an unknown length in a triangle similar to one whose lengths we know.

Example 1: Find \(x\):

\[
\begin{array}{c}
3 \\
2 \\
\end{array}
\quad \frac{3}{2} \\
4 \\
\end{array}
\]

Solution: \(\frac{3}{2} = \frac{x}{4}\) \(4 \cdot \frac{3}{2} = x\) \(6 = x\)

Similar rectangles have ratios of height to base that are equal. For example:

\[
\begin{array}{c}
1 \\
3 \\
\end{array}
\quad \frac{1}{3} \\
2 \\
6 \\
\end{array}
\]

are similar rectangles: \(\frac{1}{3} = \frac{2}{6}\).

Example 2: You wish to enlarge a 3” x 5” picture from a book when you photocopy it. If you enlarge the height (5” in the book) to 11”, what will its new base be?

Solution:

\[
\begin{array}{c}
5 \\
3 \\
\end{array}
\quad \frac{5}{3} \\
11 \\
\end{array}
\]

\(\frac{5}{3} = \frac{11}{x}\) \(5x = 33\) \(x = 6.6\) inches.
Perimeter and Area

Perimeter and area are two other important concepts in Euclidean geometry. **Perimeter** is a measurement of the distance around a shape in the plane. The units of perimeter would be things like inches, meters, or miles. **Area** measures the amount of space the shape encloses. Since measurement is basically a process of comparison, area is measured by comparing to squares where the length of each side is 1 unit. Area measurements are in units of some such standard square: square inches, square meters, square miles. As you can see in the diagrams below, the areas of the rectangles equal the number of standard squares drawn in the rectangle.

Euclid derived many formulas for areas of figures. Rather than explore these formulas, we will concern ourselves with the relationship between area and perimeter. In particular, we will focus on rectangles. For a rectangle, its area is found by multiplying its base times its height. Its perimeter is the sum of the lengths of its four sides.

Consider the following statement. Rectangles with equal perimeters can have different areas. Do you believe that? In the ancient world, many people confused area and perimeter, or even thought they were identical. In particular, many people believed that if you knew the perimeter of a rectangle, that determined (enabled you to figure out) its area, and vice versa.

Each of the rectangles below has a perimeter of 16:

A

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Area is 1 x 7 = 7

B

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2

6

Area is 2 x 6 = 12

C

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3

5

Area is 3 x 5 = 15

D

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4

Area is 4 x 4 = 16

To mathematicians, this leads to an obvious question. Of all the rectangles with a perimeter of 16, is there one that has the largest area? If there is, which one is it? In the examples above a reasonable guess would be that the 4 x 4 square at the bottom has the largest area. In true Greek style, we will prove this as a part of a more general theorem.
Theorem 1: Of all rectangles with the same perimeter, the square has the largest area.

Proof:
Start with any rectangle. We will show that the square with the same perimeter has larger area. Start with the rectangle on the left of the diagram below. Construct a square on the base of the rectangle in such a way that the square has the same perimeter as the rectangle (the length of each side of the square would be \( \frac{1}{4} \) of the perimeter of the rectangle). This construction is shown on the right. The various line segments and areas are labeled for convenience.

Because the rectangle and the square have the same perimeter, we have:
\[ 2x + 2u + 2y = 2x + 2y + 2v. \]
By subtracting 2x and 2y from each side, this implies \( 2u = 2v \), or \( u = v \). In the square, it is clear that \( x = y + v \). Thus \( vx = v(y+v) = u(y+v) = uy + ux \).

Now \( vx \) is the area B, and \( uy \) is the area A. Since \( ux \) is a positive number, B must be greater than A. So \( B + C > A + C \). That is, the area of the square is greater than the area of the rectangle. Since we started with an arbitrary rectangle, the square we constructed would have an area larger than any rectangle with the same perimeter. This concludes the proof.

Example 3: If you have a piece of string 20 inches long, find the area of the largest rectangle you can make with it.

Solution: The shape you should make is a square, according to the theorem. The length of one side would be \( \frac{20}{4} = 5 \) inches, so its area would be \( 5^2 = 25 \) square inches.

Let's extend this result about plane figures to solid figures in space, from two dimensions to three dimensions. Perimeter is a measure of the boundary of a figure in the plane; surface area is the analogous concept in three-dimensional space. Another way to tie these concepts together is to say that each measures how much stuff it takes to enclose the shape under discussion. Perimeter in the plane corresponds to how much fence it takes to enclose a field. Surface area corresponds to how much wrapping paper it takes to enclose a box (like a birthday present).

Area in the plane measures the region enclosed by the figure; volume is the analogous concept in space. For instance, the area of a field tells you how much grass it contains. Volume tells you how much cereal a box could contain. (Cereal boxes, according to a note on the box, are packed by weight, and usually don’t contain as much cereal as they could).

So a reasonable guess about a three-dimensional version of the preceding theorem would be: Of all the rectangular boxes with the same surface area, the cube has the largest volume. This turns out to be true.

Here’s a similar result about triangles. I know a calculus proof, but so far an elementary geometry proof has eluded me. Any ideas?

Theorem 2: Of all the right triangles with the same perimeter, the right isosceles triangle has the largest area.
**Example 4:** If you have a piece of string 20 inches long, find the area of the largest right triangle you can make with it.

**Solution:** The shape you should make is an isosceles right triangle, according to the theorem. All isosceles right triangles are similar (their angles are 90°, 45°, 45°).

So they all have sides proportionate to $\sqrt{2}:1:1$

![Diagram](image)

So $x + x + x\sqrt{2} = 20$

$$(2 + \sqrt{2})x = 20 \quad x = \frac{20}{2 + \sqrt{2}}.$$  

Then the area of the triangle is

$$A = \frac{1}{2} bh = \frac{1}{2} \left( \frac{20}{2 + \sqrt{2}} \right) \left( \frac{20}{2 + \sqrt{2}} \right) = \frac{200}{6 + 4\sqrt{2}},$$which is approximately 17.16 inches.

Notice that a 20 inch piece of string can enclose more area as a square (4-sided) than as a triangle (3-sided). What shape might you guess would enclose the most area?

Now consider the reverse situation in which rectangles of the same area are considered. Each rectangle below has an area of 16. For example

- **E**
  
  ![Rectangle E](image)

  Perimeter = 2 + 32 = 34

- **F**
  
  ![Rectangle F](image)

  Perimeter = 4 + 16 = 20

- **G**
  
  ![Rectangle G](image)

  Perimeter = 8 + 8 = 16
Perhaps you are now quite willing to believe the following theorem, which in a way seems to be the appropriate match to our previous theorem. Whether you believe it or not from the example, we will prove it.

**Theorem 3:** Of all the rectangles with the same area, the square has the smallest perimeter.

**Proof:**
Suppose not, i.e., assume we had a square and a rectangle with the same area, but the rectangle had the smaller perimeter. Since this assumption will be shown to be false, we can’t really draw a picture of it. But we will draw a diagram just to help keep objects clear in our mind. So square A and rectangle B have the same area, and the perimeter of B is assumed to be smaller. Now by the previous theorem, there is a square (call it C) that has the same perimeter as B, but has larger area.

![Diagram of A, B, and C]

Now let’s compare the two squares. First, since area of A = area of B < area of C, it is clear that C has the larger area. It follows that the sides of C are longer than the sides of A. However, since perimeter of A > perimeter of B = perimeter of C, A has the larger perimeter. So this would imply that the sides of A are longer than the sides of C. Clearly, both statements about the sides cannot be true. This contradiction implies that our original assumption is false. That is, we cannot have a square and rectangle with the same area where the rectangle has a smaller perimeter. The theorem is proved.

**Example:** You want to enclose a rectangular pen in your back yard for your pet pig. You want the area to be 25 square meters. What’s the least amount of fencing you will need?

**Solution:** Make the pen in the shape of a square. Then each side will be $\sqrt{25} = 5$ meters, so you will need $4 \times 5 = 20$ meters of fence.

This result can also be extended to three dimensions. In this case, it reads: Of all rectangular boxes with the same volume, the cube has the smallest surface area.

Consider a practical example: cereal boxes. Surface area is essentially the amount of cardboard it takes to make the box; volume is how much cereal can be put in the box. So if a company wants to design a cereal box to hold a certain amount of cereal, a cube would require the least amount of cardboard. Presumably that would help to reduce costs and aid the environment. From these points of view, cereal boxes shaped like cubes would seem to be desirable. They would also stack easily. Can you think of any good reasons why cereal boxes aren’t cubes?

**Integration**
Few people would doubt the significance of Euclidean geometry for its many applications to the real world. However, there are several other very important reasons why Euclidean geometry is significant.

Consider the recommendation given geometry by the great thinker Plato: “knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient... Then...
geometry will draw the soul toward truth, and create the spirit of philosophy... Nothing will be more likely to have such an effect.\(^{38}\)

For over 2000 years Euclidean geometry has been a prime demonstration of the power of reasoning. For nearly that long, Euclidean geometry has been a standard course in schools, in part for the facts one learns, but also to teach reasoning. Up until about 200 years ago, Euclidean geometry was studied by using translations and revisions of Euclid's own book! As a demonstration of the power of reasoning, Euclidean geometry became a model for the logical development of any subject matter. Aristotle modeled his approach to science after it.

St. Thomas Aquinas (1225-1274) wrote his systematic theology in the style of Euclid. He reasoned from statements he considered self-evident to prove propositions like “God exists”. For instance, consider this passage from *Summa Theologica*:

The existence of God can be proved in five ways.

The first and more manifest way is the argument from motion. It is certain, and evident to our senses, that in the world some things are in motion. Now whatever is moved is moved by another, for nothing can be moved except it is in potentiality to that towards which it is moved; whereas a thing moves inasmuch as it is moved; whereas a thing moves inasmuch as it is in act. For motion is nothing else than the reduction of something from potentiality to actuality. But nothing can be reduced from potentiality to actuality, except by something in a state of actuality. Thus that which is actually hot, as fire, makes wood, which is potentially hot, to be actually hot, and thereby moves and changes it. Now it is not possible that the same thing should be at once in actuality and potentiality in the same respect, but only in different respects. For what is actually hot cannot simultaneously be potentially hot; but it is simultaneously potentially cold. It is therefore impossible that in the same respect and in the same way a thing in the same respect and in the moved, \(i.e.,\) that it should move itself. Therefore, whatever is moved must be moved by another. If that by which it is moved be itself moved, then this also must needs be moved by another, and that by another again. But this cannot go on to infinity, because then there would be not first mover, and consequently no other mover, seeing that subsequent movers move only inasmuch as they are moved by the first mover; as the staff moves only because it is moved by the hand. Therefore it is necessary to arrive at a first mover moved by no other; and this everyone understands to be God. \(^{39}\)

As another example of the influence of Euclid, consider Spinoza (1632-1677). He wrote his Ethics with definitions, axioms, theorems and proofs. His “demonstrations” even end with the Latin abbreviation “Q.E.D.”, meaning “which was to be demonstrated” which was often used to indicate the end of proof in math books. Here are some excerpts:

**PART ONE: Of God**

**Definitions**

3 By substance, I understand that which is in itself and is conceived through itself; in other words, that, the conception of which does not need the conception of another thing from which it must be formed.

6 By God, I understand Being absolutely infinite, that is to say, substance consisting of infinite attributes, each one of which expresses eternal and infinite essence.

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Axioms

1  Everything which is, is either in itself or in another.

6  A true idea must agree with that of which it is the idea.

Proposition 8  Every substance is necessarily infinite.

Demonstration  Substance which has only one attribute cannot exist except as one substance (Prop. 5), and to the nature of this one substance it pertains to exist (Prop. 7). It must therefore from its nature exist as finite or infinite. But it cannot exist as finite substance, for (Def. 2) it must (if finite) be limited by another substance of the same nature, which also must necessarily exist (Prop. 7), and therefore there would be two substances of the same attribute, which is absurd (Prop. 5). It exists therefore as infinite substance. Q.E.D.

Proposition 11  God, or substance consisting of infinite attributes, each one of which expresses eternal and infinite essence, necessarily exists.

Demonstration  If this be denied, conceive, if it be possible, that God does not exist. Then it follows (Ax. 7) that His essence does not involve existence. But this (Prop. 7) is absurd. Therefore God necessarily exists. Q.E.D.

PART FIVE  Of the Power of the Intellect, or of Human Liberty

Proposition 17  God is free from passions, nor is He affected with any emotion of joy or sorrow.

Demonstration  All ideas, in so far as they are related to God, are true; that is to say, are adequate, and therefore (by the general definition of the emotions) God is free from passions. Again, God can neither pass to a greater nor to a less perfection (Corol. 2, Prop. 20, PART ONE), and therefore He cannot be affected with any emotion of joy or sorrow. Q.E.D.

Corollary  Properly speaking, God loves no one and hates no one; for God (Prop. 17, PART FIVE) is not affected with any emotion of joy or sorrow, and consequently He neither loves nor hates anyone.40

But Euclid’s importance does not stop with its impact on systematic theology and philosophy. Eventually, further developments in geometry had a great deal of influence on the visual art of the Renaissance. We will consider the details in a later chapter. Even the study of poetry requires some understanding of Euclid’s significance. Consider the following sonnet by Edna St. Vincent Millay:

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere

Chapter 15

In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone. 41

However you understand the meaning of the sonnet, it is obvious that you must come to grips with the sentence, "Euclid alone has looked on Beauty bare." Is Millay exalting Euclid for the beauty he alone has seen, or is she being negative in saying that the beauty Euclid saw was bare, and thereby somewhat less than ideal? We will talk more about the use of geometric concepts in literature when we discuss Flatland.

Before leaving Euclidean geometry, we ought to mention explicitly two of its limitations or weaknesses. First, Euclidean geometry is, according to Plato, about things which are "eternal", which for Plato implied unchanging. This is a weakness in that Euclid's geometry did not consider motion. Actually, some Greek philosophers went so far as to argue that motion was an illusion. The argument in one of Zeno's paradoxes went something like this: Consider an arrow flying through the air. At any instant in time, the arrow is at one particular place. At that particular time and place, it is not moving. The paradox for the Greeks was how you could take an "infinite" number of times at which the arrow was stationary, put them together, and get an arrow that was moving. (We will discuss Zeno and his paradoxes more extensively in a later chapter.) Whether or not that seems like a problem to you, Euclidean geometry did not really include motion. When the need to study motion extensively occurred during the Renaissance, Euclidean geometry needed to be augmented in order to solve the problems of the day.

Second, there is the related matter of infinity. Euclid's lines were "potentially" infinite in length, not actually infinite. That is, Euclid never imagined that his lines actually extended "to infinity" (as if infinity were an actual place). Rather, he assumed that lines were never really complete. They could always be made longer than they were at the moment, without becoming a new line. The extension was always there potentially, so it wasn't really anything new.

Perhaps this situation in geometry is related to two other aspects of the culture in which Euclid lived. First, there was the cosmos. The Greeks viewed the world around them much differently than we do today. In the typical Greek view, the cosmos consisted of a finite sphere. Lines which were actually infinite would have extended beyond the cosmos. This doesn't seem reasonable. So in Euclid's context, potentially infinite lines make sense.

Secondly, consider the typical gods of the Greeks. For the most part, they are super-humans. They have very human characteristics, but stronger or larger or faster or more passionate. But they can be viewed as mere "extensions" of humans. J. B. Phillips certainly could have said of the Greeks, "Your gods are too small!" The question is, could there be a connection between the lengths of one's lines and the "size" of one's gods? Is the rejection of the concept of actual infinity in any way connected to gods who are lesser gods than the God of the Bible?

Actual infinity has become an important part of modern mathematics. We now typically think of lines as actually infinitely long. This doesn't change the results of Euclidean geometry; it just brings it "up-to-date" with the way most mathematicians think and work in other areas of mathematics. We will discuss some of these topics again.

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41 Edna St. Vincent Millay, The Harp-weaver
CHAPTER 15: Euclidean Geometry

Homework

1. Find x for the pairs of similar triangles or rectangles.
   a. \[
   \begin{array}{c}
   3 \\
   2
   \end{array}
   \quad \begin{array}{c}
   6 \\
   x
   \end{array}
   \]
   b. \[
   \begin{array}{c}
   5 \\
   2
   \end{array}
   \quad \begin{array}{c}
   3 \\
   x
   \end{array}
   \]
   c. \[
   \begin{array}{c}
   2 \\
   3
   \end{array}
   \quad \begin{array}{c}
   3 \\
   x
   \end{array}
   \]
   d. \[
   \begin{array}{c}
   3 \\
   4
   \end{array}
   \quad \begin{array}{c}
   6 \\
   x
   \end{array}
   \]

2. Find the perimeters and areas.
   a. \[
   \begin{array}{c}
   5 \\
   2
   \end{array}
   \]
   b. \[
   \begin{array}{c}
   3 \\
   3
   \end{array}
   \]
   c. \[
   \begin{array}{c}
   4 \\
   3
   \end{array}
   \]
   d. \[
   \begin{array}{c}
   3 \\
   3
   \end{array}
   \]

3. What is the area of the largest rectangle with a perimeter of 100?
4. What is the area of the largest rectangle with a perimeter of 160?
5. If two numbers add up to 24, what the biggest that their product could be?
6. A rancher has 600 feet of fence with which to enclose a rectangular area. What's the largest area he can enclose?
7. What is the smallest perimeter a rectangle can have if it has an area of 100?
8. A rancher wants to enclose a rectangular area of 400 square feet. What's the smallest amount of fence he could use?
9. If the product of two numbers is 64, what is the smallest sum they could have?
10. If the sum of two numbers is 26, what is the largest their product could be?

11. A rancher has 600 feet of fencing with which to enclose a right triangular area. What is the largest area which he can enclose?

12. You have 500 feet of fencing which you want to use to enclose a right triangular area. What is the largest area which you can enclose?

Selected Answers:

1.  a. 4 b. 7.5 c. 4.5 d. 4.5
3.  625 (a 25 by 25 square)
5.  144 (12 and 12)
6.  22,500 square feet
7.  40
8.  80 feet (a 20 foot by 20 foot square)
9.  16 (8 and 8)
10. 169
11. 15,441.6 square feet