CHAPTER 18: Mathematics and Science

It is not possible to have a reasonable belief against miracles.\textsuperscript{87}

Since long ago, people have wondered about the world around them. The Babylonian view was that nature was divine and was to be feared; nature has a life of its own and is capricious. With such a belief, attempting to find a pattern associated with the events of the universe is futile; prediction of future events is impossible, because there is no basis upon which to expect regular behavior. On the other hand, the Babylonians did observe some regularity in the motion of heavenly bodies.

The Hebrew view was that God made nature; it is not divine, but quite distinct from the Creator. He alone is to be feared. Nature for its part is orderly, because God who is a God of order has ordained it to be so. Miracles can and do occur; these are events in which God chooses to act in a way which is different than His usual "predictable" way. As Pascal notes, this is entirely consistent with God's nature as God.

The Greek view was that the universe was rational (indeed, mathematical) and “living”. In its rationality, the Greek view comes close to the Biblical view, except that the rationality of the Greeks is a principle, not the result of the work of a personal God. In suggesting that the universe is "living", the Greeks meant that it is animated in itself, as opposed to being matter created by the Living God. The Pythagoreans had two key doctrines: nature is built in accordance with mathematical principles and number relationships reveal the order in nature.\textsuperscript{88} Christians can accept these views with the understanding that the principles and relationships are the result of God's creative work. The Biblical view offers the explanation for the parts of the Greek view which are true.

Science is the systematic attempt to understand nature. It assumes that nature is orderly, and that we can discover some of that order. Both of these assumptions are justified by the doctrine of Creation.

One particular area of Greek science which is important for us is their understanding of the earth and the heavens. The basic assumption was that the heavens and the earth are two different domains. That is, the matter which made up the heavenly bodies like the sun, the moon and the stars was different than the matter which made up the earth. Motion on earth had nothing to do with how the sun and moon moved.

The astronomer to whom the basic Greek scheme of the universe is attributed is Ptolemy, who lived around 150 AD. Ptolemy taught that the earth is the center of the universe and does not move. The sun and moon obviously moved in circular paths across the sky. To describe the motion of the planets, Ptolemy also used circles. Unfortunately, planetary motion as we observe it is not as simple as the motion of the sun and moon: some planets actually seem to move backwards! To explain this, Ptolemy used a system of epicycles, circles moving on circles. The net result was rather accurate: the motions of the heavenly bodies were described by the Ptolemaic system as accurately as they could be observed at the time (in fact, as accurately as they were observed until the seventeenth century). The other nice thing about Ptolemy's system was that it was aesthetically pleasing. Yes, a type of beauty is a criteria for a scientific theory, and the use of only perfect circles had that kind of appeal for the Greeks. In fact, Ptolemy's system was so appealing that it was the accepted understanding of the universe for 1500 years.

From 500 A.D. to 1000 A.D., Western Europe went through a period often called “The Dark Ages”. Culture in general, and mathematics in particular, stagnated. Much of the classic learning of the Greeks was discarded or forgotten. The Church was the most significant social institution. Concern for the salvation of one's soul in the afterlife was of prime concern. Day-to-day living was difficult for almost everyone; there was no leisure time to spend on abstract studies such as mathematics. Theology as it appeared in Scripture interpreted by the Church Fathers -- that was all the knowledge or "science" one needed to know. Apart from the arithmetic of everyday commerce, mathematics appeared to have little use

\textsuperscript{87}Pensees, p. 284.
\textsuperscript{88}Kline, Mathematics for Non-Mathematicians, p. 189.
with two notable and ancient exceptions: the development of the calendar (with prime concern about the date of Easter) and astrology.

**Arabic Mathematics**

Arabic culture in conjunction with Islamic thought went through two stages in its reaction to European civilization. At first, during the early years of Islam (the seventh century AD), a religious fundamentalism reacted negatively and strongly to the civilization associated with Christianity. During the Muslim conquests in the Mediterranean area, Greek mathematical texts along with other elements of western culture were destroyed. However, after a time this fanaticism gave way to an appreciation of learning. In fact, it is to the Arabs that western Europe owes the preservation of Greek mathematics during the European Dark Ages. The Arabs grew in their appreciation of the developments of Thales, Pythagoras, Euclid, Archimedes, and others. At first, they merely preserved what they found, translating it into Arabic. Then they began to write extensive commentaries on the Greek texts. In arithmetic, they rejected negative numbers, but treated irrationals as numbers, perhaps under influence from the Hindu culture to the east. Recall that we have seen that the Hindus had discovered how to do arithmetic with radicals, a feat which had eluded the Greeks.

Mathematics in Arabic culture had similar motivations to other cultures. There were the needs of astronomy, which was applied to the practical problem of determining exact times for prayer. From a science point of view, power over nature, not understanding for its own sake, seemed more influential. And there were the practical matters such as inheritance and commerce. But there was also a curiosity about mathematics for its own sake.

A significant turning point occurred during this period of time as the Arabs reflected on mathematics and the subject was lying dormant in Europe. Arithmetic made serious steps toward becoming algebra. While forces internal to mathematics were certainly pushing in this direction, it is also true that the surrounding culture proved conducive to the development. "It can be said that the step from arithmetic to algebra is in essence a step from "being" to "becoming" or from the static universe of the Greek to the dynamic, ever-living, God-permeated one of the Muslims." The suggestion here is that the Greek view of the universe lent itself to the study of geometry, the study of the properties of objects that are unchanging, whereas the Muslim view that incorporates an active Deity provides impetus for a mathematics of "variables" as in algebra. Can the dominant theology and view of the nature of the universe influence the development of its mathematics? Yes, it certainly can. By the way, the book Al-Jabr wa-al-Muqabala by Al-Khwarizmi, written in 820 AD is the source of the word "algebra".

Omar Khayyam, best known to most people for his poem, the "Rubaiyat", was also a mathematician. He lived around 1100 AD. He, along with his fellow Arabs, appreciated the logical rigor of Greek geometry. Omar Khayyam was one of many mathematicians to explore the parallel postulate of Euclid; his work helped pave the way for non-Euclidean geometries. His algebra was geometric, following the approach of the Greeks, including Euclid. He solved cubic equations by the use of intersecting conic sections (circles, parabolas, ellipses and hyperbolas); the Greeks had solved quadratic equations by similar methods. This was a significant advance on which European mathematicians would later build.

**Western Europe after the Dark Ages**

By 1000 A.D., there was increasing contact between the peoples of western Europe and the Arabs and Arabic learning. The Arabs had preserved much of classical Greek learning. It had been translated into Arabic, studied, commented upon, and in some cases improved or expanded. Now came the translation of Greek learning from Arabic into Latin, the language of the scholar in western Europe. During the next two hundred years, bit by bit, western Europe became reacquainted with its heritage.

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It was during this period that the first universities appeared in western Europe. Scholars had groups of students join them in exploring the works of the Greeks and Arabs. The revival of Greek learning did not meet with universal acceptance. For instance, the study of Aristotle was banned at University of Paris in 1210; the penalty was excommunication. The Church was not about to yield any of its authority to Aristotle.

The revival of classical Greek mathematics did bring with it a use for the study of mathematics. For Plato, the study of mathematics had been the necessary preparation for the study of philosophy. In these centuries this was adapted so that mathematics became a preparation and model for the study of systematic theology. In Anselm’s ontological argument we see an almost Euclidean concern for proof from basic concepts and self-evident truths. In Abelard’s Sic et non ("Yes and no"), he collected statements from the Church Fathers which contradicted each other and attempted to develop a logically consistent system which would eliminate the contradictions. Abelard was trying to do for the Church's theology what Euclid had done for geometry.

A significant person in this period is St. Thomas Aquinas (1224 –1274 A.D.). Aquinas developed theology systematically. His organization of Catholic doctrine was foundational for centuries. There are two reasons why Aquinas is an important figure in our story. The first is very broad and general. In medieval times, the Church held that knowledge was gained from authorities. One knew what one knew because it was revealed: the Scriptures and the Church Fathers taught it as God had revealed it. With Aquinas, the Church now lent its support to the notion that reason is a reliable source of knowledge: truth can be deduced from revelation. Aquinas organized and wrote his theology the way Euclid had done geometry. Instead of depending solely on revelation to obtain knowledge, people now began to trust also in reason. While this is certainly a broad generalization, there is perhaps a general trend which is worth noting. The following chart suggests a general cultural trend in Europe.

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>HOW KNOWLEDGE WAS OBTAINED</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000-1200</td>
<td>revelation</td>
</tr>
<tr>
<td>1200-1500</td>
<td>revelation primary, reason secondary</td>
</tr>
<tr>
<td>1500-1700</td>
<td>reason primary, revelation secondary</td>
</tr>
<tr>
<td>1700-Present</td>
<td>reason</td>
</tr>
</tbody>
</table>

The second reason Aquinas is important to us is another instance of a type of integration of faith and learning (recall the Pythagoreans). In this case, Aquinas attempted an integration of Aristotle's philosophy and Catholic doctrine. This is important because it puts the Church on the side of Aristotle's method and conclusions with regard to science.

Up until this time, the Church basically saw nature as God's orderly creation with the order revealed in Scripture. Since a distinction will now become important, revelation in Scripture is referred to as special revelation. For the medieval church, the Bible taught everything people needed to know about the world in which they lived (although the focus of the Bible was of course on the much more important world to come).

Now Aristotle in his discussion of knowing had distinguished four different "causes" or types of "explanation" of a phenomena.

1) First was the efficient cause; it answered the question, "who did it?"
2) Second was the final cause, or end in the sense of purpose. (Following the Greek word telos, this is sometimes called the teleological cause.) This cause answered the question, "why was it done?"
3) Third was the material cause; "what is it made of?"
4) Finally, there was the formal cause: "how was it done?" 

As a relevant illustration, consider the creation of the world. The efficient cause would be the person responsible for creation, God. The final cause might be to glorify Himself. The material cause would be matter and energy: chemical elements and physical laws. Then there's the formal cause: a literal six day out-of-nothing, just by speaking explanation or some sort of theistic evolutionary scheme? I claim that the main thesis of Genesis 1 is not concerned with the details of how God made the world (its formal cause); the main thesis of Genesis 1 is that God Himself made it. Well, without attempting to answer the last question (this is, after all, a math class, not a Bible class or a science class), at least the different types of causes should be clearer.

Now for Aristotle and Aquinas, the efficient and final causes were of utmost importance. Aristotle and Aquinas saw the person and the purpose as most important for explanations. And, very significantly, this was their conception of what science was all about. So when followers of Aquinas discussed the scientific explanation for the universe, they appealed to the authority of Scripture to say that God did it for His glory, and the discussion was over. Well, a shift was coming. Aristotelian science would be challenged by a new science (what we will call "modern science"). For the new scientists, science would no longer concern itself with who or why, but rather what and how. Although it is somewhat simplistic, theology was allowed to discuss the who and why questions while science went on to the issues of what and how, and the modern sciences of chemistry, physics, and biology were born.

But that is to move just a bit ahead in the story of mathematics. It would take several centuries for the sparks of modern science to burst into flame, fueled by the developments of mathematics in those centuries. The period from 1200 to 1500 A.D. is known as the Renaissance.

There were a number of aspects of the general development of civilization in western Europe in the Renaissance which affected and were affected by mathematics. For instance, Hindu–Arabic numerals, including 0, had come to western Europe in the 1200's. While it took quite a while for them to be accepted over the more familiar though inferior Roman numerals, their availability was significant. The compass was borrowed from the Chinese; it provided a tool necessary to expand navigation, which in turn provided new problems for mathematics. From the Chinese also came gunpowder. In the hands of the peoples of western Europe, that led to a number of questions about the motion and trajectories of projectiles, namely cannonballs. The development of moveable type for the printing press increased dramatically the accessibility of knowledge. The telescope and microscope were developed, and both depended on sophisticated knowledge of the curves used in grinding lenses. And the telescope in particular became the essential tool of astronomy, an old ally of mathematics. Last, but not least, there is Renaissance art, which shall be discussed in the next chapter.

The Rise of Modern Science

In the Renaissance, a strange sort of integration led to the beginnings of modern science. Morris Kline describes the blending of ideas this way:

"Thus Catholic emphasis on a universe rationally designed by God and the Pythagorean–Platonic insistence on mathematics as the fundamental reality of the physical world were fused in a program for science which in essence amounted to this: science was to discover the mathematical relationships which underlie and explain all natural phenomena and thus reveal the grandeur and glory of God's handiwork." 

Here we see two key ideas: the Greek idea that the universe is fundamentally mathematical, and the Christian idea that the Creator revealed something of Himself in His creation.

Richard Westfall adds a third idea to this mix:

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91W.D. Stangl, "The Language of Creation", unpublished manuscript.
92Morris Kline, Mathematics in Western Culture, p. 108.
“Two major themes dominated the scientific revolution of the 17th century — the Platonic-Pythagorean tradition, which looked on nature in geometric terms, convinced that the cosmos was constructed according to the principles of mathematical order, and the mechanical philosophy, which conceived of nature as a huge machine and sought to explain the hidden mechanisms behind phenomena.”

The third idea is the contribution of the Renaissance: the universe is like a machine (rather than like a living organism, for instance).

These were the principles that provided the groundwork for modern science to be built: a rational Creator made a "mechanism" based on mathematical principles which were capable of being understood by His highest creation, human beings, made in His image.

<table>
<thead>
<tr>
<th>Foundations of Modern Science</th>
</tr>
</thead>
<tbody>
<tr>
<td>The universe is ...</td>
</tr>
<tr>
<td>made by God</td>
</tr>
<tr>
<td>machine-like</td>
</tr>
<tr>
<td>mathematically designed</td>
</tr>
</tbody>
</table>

Our first example of the results of modern science is the Copernican revolution. Copernicus lived around 1500, the time of the Reformation. Following Ptolemy, Copernicus continued to believe that planetary motion should be in circular paths and that planetary motion should be at a constant velocity. However, after considerable reflection on the rather extensive observational data available to him, much of it from Arabic sources, Copernicus proposed a new theory. It was called the heliocentric theory, because it placed the sun, not the earth, at the center of the solar system. The system of Copernicus, however, continued to be a complex conglomeration of circles and epicycles.

It was Johannes Kepler who found the mathematical keys to a simpler system. Kepler was trained in part for the Lutheran ministry. He was a firm believer in the God of the Bible as the Creator of the cosmos. He was also a bit of a Pythagorean. He believed that God would have used an elegant and simple plan based on numbers to design the universe. When he wrote the results of his life's work in 1619, he entitled the book The Harmony of the World. By "harmony", he did not mean simply that the cosmos was orderly; he literally was thinking about "the music of the spheres [planets]."

Kepler used a combination of mathematical and musical intuition, and a large number of observations through the newly-discovered telescope, to develop three famous laws of planetary motion. The process was not an easy one, because Kepler in many ways was a product of his times and still thought like the Greeks. He assumed God would have used circles to design the universe because circles were the simplest and most perfect curves. For years he tried to make sense of his observations using circles as the orbits of the planets. In his first law, Kepler abandoned the circle and epicycles as the path of a planet. In their place, he found that a single ellipse would work perfectly.

In his second law, Kepler abandoned the ancient belief that planets always traveled at a constant velocity. He found a formula which described a variable velocity. When the planet was close to the sun, it traveled faster than when it was further from the sun.

Finally, Kepler searched for a numerical pattern in the locations of the planets. He was certain that God had arranged the solar system in some "harmonious" pattern. Finally, he came across the law: if $T$ is the period of revolution of any planet (how long that planet takes to go around the sun once, its "year"), and $D$ is its average distance from the sun, then the square of the period is proportional to the cube of the distance. This is written algebraically

$$ T^2 = c \ D^3. $$

We will work with this in the form it appears when you take the square root of both sides:

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where \( k \) is a constant which is the same for all the planets in our solar system. We can determine \( k \) if we know both \( T \) and \( D \) for one planet. For instance, we can use what we know about the earth: \( T = 1 \) year and \( D = 149.6 \) million kilometers. So

\[
1 = k (149.6)^{1.5}
\]

or

\[
k = \frac{1}{149.6^{1.5}} = \frac{1}{149.6 \sqrt[1.5]{149.6}} = 0.0005465.
\]

**Example 1:** The average distance from the sun to the planet Neptune is 4497 million kilometers. How long would it take for Neptune to complete one revolution around the sun?

**Solution:**

\[
T = k D^{1.5} = (0.0005465) D^{1.5} = (0.0005465) (4,497)^{1.5}
\]

\[
= (0.0005465) (4,497) \sqrt[1.5]{4497} = (0.0005465) (301,567)
\]

\[
= 165 \text{ years.}
\]

While Kepler discovered some fundamental laws of the new astronomy, he also continued to accept some ancient and medieval views. For instance, he wrote horoscopes, although he rejected many of the more fantastic ideas connected with astrology.

As an astronomer, Kepler's goal was to make 'as many discoveries as possible for the glorification of the name of God and sing unanimous praise and glory to the All-wise God.'\(^{94}\) Kepler's intention was clearly to understand the creation more fully so that its Creator could be more accurately praised.

"Since we astronomers are priests of the highest God in regard to the book of nature, it befits us to be thoughtful not of the glory of our minds but rather, above all else, of the glory of God."\(^{95}\) Kepler refers to the "book of nature." This was a common name at that time for the physical universe. God's revelation of Himself was seen to be two-fold: first, His written revelation in the book of Scripture, the Bible; and second, His revelation through His creation and providential care of the physical universe. Since some of his characteristics are evident in what He has made (Romans 1: 20). So the scientists of the 1600's referred to the universe as a "book" analogous to the Bible. They frequently saw themselves as simply studying the book of nature the same way theologians studied the book of Scripture, trying to discover and display what they could about God.

The Greeks often viewed the universe itself as divine and living. For Kepler, it was important to distinguish the Creator from his creation. In part this difference was reflected in the use of the language of the machine. "My goal is to show that the heavenly machine is not a kind of divine living being but similar to a clockwork in so far as almost all the manifold motions are taken care of by one single absolutely simple magnetic bodily force, as in a clockwork all motion is taken care of by as simple weight."\(^{96}\) [You need to remember that clocks in Kepler’s day typically used a hanging weight to keep them running, not a spring or a battery.]

Many of Kepler's contemporaries around 1600 contributed to the development of the new scientific method and ushered in the scientific revolution. One very prominent contributor was Galileo (1564–1642). Galileo expressed the characteristics of modern science this way:

1) First, science doesn't explain why, rather it describes how. In Aristotle's terms, science does not seek teleological explanations; rather, it seeks material and formal causes.

\(^{94}\)quoted in Kepler, Max Caspar, translated by Doris Hellman, Dover Publ., New York, 1985, p. 64.

\(^{95}\)quoted in Kepler, p. 88

\(^{96}\)quoted in Kepler, p. 136.
2) Second, there was a revolt against the authority of Aristotle and the Church. Galileo proposed to look at nature itself, directly, to discover how it was made. Galileo argued that God was as much the Author of the Book of Nature as He was of the Book of Scripture. Indeed, natural theology was information about God which one could gain from the study of the Book of Nature.

3) Third, modern science had new motivations. Even though the rise of modern science was a revolt against the authority of Aristotle and the Church, it was not a revolt against God. The early modern scientists were almost all firm believers in the God of the Bible. Their motivations were hence still Christian in nature. One motivation for the study of nature was to worship the Creator, and another was to learn how to utilize the world God had given to human beings to care for.\(^{97}\)

Galileo's methodology was based on mathematics in several ways. First was the issue of the fundamental principles of science. The Greeks had believed that science began with fundamental principles. However, they either believed that these principles were innate (in our minds when we are born) or that these principles were capable of being discovered by rational thought. Aquinas took over the belief in fundamental principles, but said that they came from Scripture. Galileo disagreed -- he said that the fundamental principles would come from observation of nature itself, either directly or through those controlled observations we call experiments. The fundamental principles would concern observable entities, and the observations would be numbers.

To understand how revolutionary this concept was in Galileo's time, it would be useful to consider a concern raised by the Greek philosopher Zeno. Zeno had believed that really real things have permanence, and so change is a bit of an illusion. Since motion is change in position, Zeno didn't believe that real things move. If that seems just a bit weird to you, just go along with the idea for a few minutes.

In fact, Zeno had formulated one of his paradoxes against the concept of motion by considering essentially the idea of instantaneous velocity, although he didn't call it that. Zeno considered the flight of an arrow; you may think of some other object if you wish. At an instant in time, what is the arrow doing? In modern terms, if you were to take a picture of the arrow, what would you see? Zeno claimed that what is happening in an instant in time is that the arrow is at one particular place, and therefore is not moving. Again, in modern terms, perhaps Zeno would suggest that the camera captures the motionlessness of the arrow in the picture you take. Zeno's argument, since he didn't have a camera, was something like this: if the arrow was moving at that instant in time, it would need to be in one place at one time in the instant, but at another place at another time in the instant. But then the instant in time would have subdivisions, but you can't divide an instant into parts, or it wouldn't be an instant.

So Zeno had determined that at any particular instant in time the arrow was not moving. Now the time duration of the flight of the arrow consists of instants in time. If at each instant in time, the arrow is not moving, then it can't be moving over the entire time of the flight: how could motion arise out of instants of non-motion?! Motion is an illusion.

Now on first glance, this sounds preposterous - I suspect we all believe arrows really move when they fly. But what is the problem with Zeno's analysis of the situation? That is the paradox. The more you think about it, the more puzzling it becomes. Centuries of questions were the context for Galileo's attempt at clarifying the issue.

Galileo proposed a new analysis of motion which presumed that he could identify a speed with such an arrow at an instant in time. Zeno would have been shocked! When we find an instantaneous velocity for a falling rock, we are claiming that at an instant in time, the rock is in some sense really moving, and we know how fast! It's instantaneous velocity is not something which we observe (as in a picture), but rather something which we deduce. What we see in a picture of a moving object is where it is located at an instant in time, but we can't see its instantaneous velocity. With this insight, what Galileo presented was a coherent account of motion.

Galileo's fundamental concepts include distance, time, speed (both average and instantaneous), weight, mass, and force, some of which are quite abstract. Distance is easy to measure. On the other hand,

\(^{97}\)Genesis 1:26,28; 2:15.
time was very difficult to measure in 1600. “Within three decades of Galileo’s death the average error of the best timepieces was reduced from fifteen minutes to only ten seconds per day.”

Average velocity is easy to compute once you know distance and time, but instantaneous velocity is another matter. We will discuss this concept shortly. Before Galileo, forces were understood to always involve contact. When a rock fell, it was being pulled or pushed by something material. After Galileo and Newton, “force” could include "action at a distance", meaning that no contact was involved. Gravity was such a force.

Once the fundamental principles were established, Galileo proceeded in the manner of Euclid: deduction and mathematical analysis. We will illustrate this type of approach below.

Consider the motion of a rock dropped from a tall building. Galileo was interested in a mathematical description of what happens. Imagine that he was able to make the following observations, where \( t = \) elapsed time (in seconds) from when the rock was dropped, and \( s = \) distance (in feet) the rock has fallen in time \( t \):

<table>
<thead>
<tr>
<th>( t )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>64</td>
</tr>
<tr>
<td>3</td>
<td>144</td>
</tr>
<tr>
<td>4</td>
<td>256</td>
</tr>
</tbody>
</table>

Galileo was searching for a pattern, a formula connecting \( t \) and \( s \). Perhaps he made the following observations, one column at a time:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( s )</th>
<th>( s )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>16 • 0</td>
<td>16 • 0(^2)</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>16 • 1</td>
<td>16 • 1(^2)</td>
</tr>
<tr>
<td>2</td>
<td>64</td>
<td>16 • 4</td>
<td>16 • 2(^2)</td>
</tr>
<tr>
<td>3</td>
<td>144</td>
<td>16 • 9</td>
<td>16 • 3(^2)</td>
</tr>
<tr>
<td>4</td>
<td>256</td>
<td>16 • 16</td>
<td>16 • 4(^2)</td>
</tr>
</tbody>
</table>

Now the connection between \( t \) and \( s \) is apparent. If there is a pattern, it certainly looks like the formula

\[
\boxed{s = 16 \cdot t^2.}
\]

We can easily use this formula to find a formula for the average velocity of the rock from the moment it is dropped until any later time. If \( t \) represents the total time the rock is falling, then

\[
\text{average velocity} = \frac{\text{total distance}}{\text{total time}} = \frac{s}{t} = \frac{16 \cdot t^2}{t} = 16 \cdot t.
\]

For instance, over the first 3 seconds of the rock’s fall, it travels a distance of 144 feet, and it’s average velocity would be \( 16 \cdot 3 = 48 \) feet per second.

We can also use the formula to find the average velocity over any time interval. For instance, suppose we want the average velocity of the rock from time $t = 1$ second to time $t = 3$ seconds. How would we find the total distance traveled in this time interval that lasts 3 seconds? The total distance traveled is the difference between how far the rock traveled in 3 seconds and the distance traveled in 1 second. Thus we have

$$
\text{average velocity} = \frac{\text{total distance}}{\text{total time}} = \frac{(16 \cdot 3^2) - (16 \cdot 1^2)}{3 - 1}
$$

$$
= \frac{144 - 16}{2} = 64 \text{ feet per second.}
$$

Galileo was also interested in a much more difficult concept, instantaneous velocity. Instantaneous velocity is how fast the rock is moving at any instant in time. In our everyday experience, that is roughly what your speedometer tells you about your car. Conceptually, this is very different from average velocity. There is no time interval over which to observe the object moving. We are talking about an instant in time, not just a small time interval. However, average velocity seems to be the only concept we have from which to develop the concept of instantaneous velocity.

Consider the following example in which we imagine Galileo attempting to discover the instantaneous velocity of the rock at the instant when $t = 3$ seconds. His method is to observe how far the rock travels in shorter and shorter intervals of time around the instant when $t = 3$. Then he calculates the average velocity over these time intervals. His observations and calculations are summarized in the following chart:

<table>
<thead>
<tr>
<th>first t</th>
<th>first s</th>
<th>second t</th>
<th>second s</th>
<th>distance</th>
<th>time</th>
<th>average velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>64</td>
<td>4</td>
<td>256</td>
<td>192</td>
<td>2</td>
<td>96</td>
</tr>
<tr>
<td>2.5</td>
<td>100</td>
<td>3.5</td>
<td>196</td>
<td>96</td>
<td>1</td>
<td>96</td>
</tr>
<tr>
<td>2.9</td>
<td>134.56</td>
<td>3.1</td>
<td>153.76</td>
<td>19.2</td>
<td>.2</td>
<td>96</td>
</tr>
</tbody>
</table>

To observe any shorter time interval in Galileo’s day would be ridiculous to consider. Fortunately, we already see a pattern, and algebra is all that is needed to verify the results in general. First, we illustrate the algebra for any interval centered around the instant when $t = 3$:

average velocity from time $3 - h$ to time $3 + h$

$$
= \frac{16 (3 + h)^2 - 16 (3 - h)^2}{2h} = \frac{16 (9 + 6h + h^2) - 16 (9 - 6h + h^2)}{2h}
$$

$$
= \frac{144 + 96h + 16h^2 - 144 - 96h - 16h^2}{2h} = \frac{192h}{2h} = 96.
$$

The idea is now to suggest that if the average velocity over any such interval around the instant when $t = 3$ seconds is 96 feet per second, even very, very short time intervals, then the only reasonable value for the instantaneous velocity at the instant when $t = 3$ is also 96.

To derive a formula for instantaneous velocity at any time $t$, we simply repeat the above procedure with a "$t$" in place of the "$3$".
average velocity from time $t - h$ to time $t + h$

\[
\frac{16 (t + h)^2 - 16 (t - h)^2}{2h} = \frac{16t^2+32th+16h^2 - 16t^2 -32th -16h^2}{2h} = \frac{64h}{2h} = 32t.
\]

Thus we have mathematically derived the formula

$$v = 32t$$

which gives instantaneous velocity ($v$) at the (instant in) time ($t$).

To illustrate what Galileo was able to do once he had derived such a formula, consider the following question. How is the instantaneous velocity of a dropped rock related to the distance it has fallen? Aristotle believed that the rock's speed increased in direct proportion to the distance it fell. This simply seemed reasonable: if the rock fell four times as far, it would be going four times as fast.

Galileo used algebra to find the formula relating distance ($s$) and instantaneous velocity ($v$). First, he solved the equation $s = 16t^2$ for $t$:

\[
\frac{s}{16} = t^2 \quad \text{and} \quad \sqrt{\frac{s}{16}} = t.
\]

Now substitute this formula for $t$ into the formula for $v$:

$$v = 32 \sqrt{\frac{s}{16}} = \frac{32}{\sqrt{16}} \sqrt{s} = 8 \sqrt{s}.$$

This formula indicates that the rock's speed increases in proportion to the square root of the distance it fell. To everyone's surprise, it turned out that if the rock fell four times as far, it would only be going twice as fast!

We have seen that Kepler discovered basic laws concerning the motion of heavenly bodies. For the most part, these were ingenious descriptions of observations made over the years. Galileo had discovered laws concerning motion of bodies here on earth. For his part, Galileo had not only done observations, but had derived his descriptions of motion from some fundamental laws. But did Galileo's laws have any relation to Kepler's descriptions? There was good reason to not even think of such a question. After all, the heaven and the earth were viewed as two different realms. Common thought assumed they contained two different kinds of bodies, made of different material and obeying different laws. But now that the Copernican heliocentric theory had suggested that the earth is just one of the planets, a person could begin to wonder...

The last person we will mention in this chapter is Isaac Newton, one of the greatest mathematicians and scientists ever. Newton was born in the year that Galileo died (1642). For our purposes at the moment, we will consider only one of Newton's contributions, his famous law of universal gravitation. The key word is "universal". It was Newton's crowning achievement to discover that the laws of motion are the same in both the heavens and the earth; the "law of gravitation" is universal in the sense that it is the same throughout the universe! Newton used this universal law to explain why the planets move in orbits that are ellipses, as well as explaining the other laws of Kepler. He also showed that the laws of Galileo that explained falling rocks (and falling apples!) and the trajectories of cannonballs here on earth explained the
orbit of the moon around the earth (it's like a fast-moving cannonball). And he showed that heavenly bodies and the earth "interact"; for instance, the moon "governs" the tides on earth by the law of universal gravitation. Newton provided the grand synthesis that showed us that the cosmos is a universe; the heavens and the earth are all part of one integrated system, all created and governed by God.

Speaking of integration, much has been written about the relationship between Newton's belief in God and his science. It is certain that Newton believed that God created and sustains the universe by His power, and there is strong evidence that Newton desired that his scientific accomplishments point to the glory of the Creator. (For instance, I've written "Mutual Interaction: Newton's Science and Theology", *Perspectives on Science and Christian Faith*, Vol. 43, #2, June, 1991, pp. 82-91.)

**TOPIC 18: Mathematics and Nature**

**Homework**

1. Find the number of years in one revolution around the sun for each of the planets listed. The average distance from the sun is provided in millions of kilometers.
   a. Mercury $D = 57.9$
   b. Venus $D = 108.2$
   c. Saturn $D = 1427$

2. Find the number of years in one revolution around the sun for each of the planets listed. The average distance from the sun is provided in millions of kilometers.
   a. Uranus $D = 2,869$
   b. Mars $D = 228$
   c. Jupiter $D = 778$
   d. Pluto $D = 5,900$

3. A rock is dropped. Find how far it has fallen after
   a. 2 seconds
   b. 4 seconds
   c. 6 seconds

4. A rock is dropped. Find its average velocity
   a. over the first 3 seconds of its fall.
   b. from $t = 1$ second to $t = 4$ seconds
   c. from $t = 3$ seconds to $t = 5$ seconds
5. A rock is dropped. Find its instantaneous velocity at
   a. $t = 2$ seconds
   b. $t = 4$ seconds
   c. $t = 5$ seconds

6. A rock is dropped. How long does it take for it to fall
   a. 64 feet
   b. 256 feet
   c. 550 feet
   d. 1000 feet

7. A rock is dropped. How fast will it be moving (instantaneous velocity) when it has fallen
   a. 64 feet
   b. 256 feet
   c. 550 feet
   d. 1000 feet

Selected Answers

1. a. 0.24 years  b. 0.62 years  c. 29.4 years
3. a. 64 feet  b. 256 feet  c. 576 feet
4. a. 48 feet/sec  b. 80 feet/sec
5. a. 64 feet/sec  b. 128 feet/sec  c. 160 feet/sec
6. a. 2 seconds  b. 4 seconds
7. a. 64 feet/sec  b. 128 feet/sec