CHAPTER 20: Coordinate Geometry

This is our true state; this is what makes us incapable of certain knowledge and of absolute ignorance. We sail within a vast sphere, ever drifting in uncertainty, driven from end to end. When we think to attach ourselves to any point and to fasten to it, it wavers and leaves us; and if we follow it, it eludes our grasp, slips past us, and vanishes for ever. Nothing stays for us. This is our natural condition, and yet most contrary to our inclination; we burn with desire to find solid ground and an ultimate sure foundation whereon to build a tower reaching to the Infinite. But our whole groundwork cracks, and the earth opens to abysses.

Let us therefore not look for certainty and stability. Our reason is always deceived by fickle shadows; ...

Many people during the Renaissance found that the certainty accorded by the medieval world view in Europe was no longer available to them. The answers which had been provided so authoritatively were no longer accepted without question, and in some cases they had been shown to be wrong. Aristotle had been wrong about motion, Ptolemy was wrong about the solar system, the Roman Catholic Church was wrong (according to Luther) about salvation, and Luther was wrong (according to the Anabaptists) about infant baptism. Newton's universal law of gravitation explained a great deal, but the net result of the heliocentric theory and Newton's work was that the earth became a very tiny speck in the universe, moving along with everything else, and nowhere near the center of things. Even though most people still believed in God, it was hard not to think that our place in the broad scheme of things was much less significant.

This is the context into which Pascal spoke. His first quote above reflects the feelings of many who felt adrift in the vastness of space. If there was only some solid ground, some certainty. Pascal saw some philosophers of his day trying to find this certainty through reason. While Pascal certainly valued the gift of reason, he saw human reason as unequal to the task of finding certainty. Certainty, too, was a gift, from God and by faith.

A contemporary and acquaintance of Pascal who sought certainty through reason was Rene Descartes. Descartes is considered by many to be the father of modern philosophy. A man with a wide range of interests and intellectual pursuits, Descartes studied mathematics and discovered in it the certainty he longed for in other areas. In the axiomatic method, Descartes found the method by which certain knowledge could be obtained. In a time when the knowledge of the past in almost every discipline was being called into question or changed, Euclidean geometry stood firm.

In his Discourse on Method, Descartes presented and applied his plan for the search for certain truth. He first needed axioms, those self-evident truths to which any rational person would give assent. The first of these was his famous, "I think, therefore I am." Other axioms asserted that each phenomena must have a cause, an effect cannot be greater than its cause, and the mind has within it the concepts of perfection, space, time, and motion. From the axioms, Descartes used deductive reasoning to prove theorems, such as "God exists". Descartes' plan is clearly an example of how significant human reasoning had become in European culture, and of how much influence mathematics had in that development.

Descartes is also seen by many as the father of modern mathematics. Until 1600, geometry had been the dominant branch of mathematics in Europe. This was mostly the legacy of the Greeks, who had been unable to produce a clear and consistent theory of numbers other than rational numbers. This slowed the development of algebra in Greek thought. Arabic influence had given impetus to the development of algebra, and set the stage for the contribution of Descartes.

While Euclidean geometry had certainly been a grand success in proving a large number of theorems, it did have some weaknesses. Euclid's approach was often tied to the construction of diagrams with restrictive tools. Euclid used techniques in his proofs that often worked only in very special situations. The emerging scientific studies were leading to new curves not easily examined by Euclid's methods.

102Pascal, Pensees, p. 29.
Numerical information about curves was needed. These conditions challenged mathematicians to develop a new approach to geometry. Descartes and others produced a synthesis of geometry with algebra. The newly-emerging power of algebra became available for the solution of geometry problems.

**Cartesian coordinate system**

The Cartesian coordinate system, named after Descartes, is the basis for the algebraic approach to geometry. We start with a single horizontal number line. (Actually, there is no reason why we couldn't start with some other line, but we will stick with a horizontal line for convenience.) A number line is a geometric object (a line consisting of points) combined with arithmetic (each point on the line is associated with a real number). At first, you might think of a ruler, and wonder what's so complicated about this idea. However, the next step should alert you to the difficulties encountered in Descartes' time.

The number 0 is placed in the middle of the number line, with the positive numbers to its right. In the 1500's in western Europe, 0 was not that familiar a concept. (Have you ever seen a ruler with a "0" label on it?) That leaves the left side of the line for the negative numbers. Some early attempts assumed that the negatives would "line up" just like the positives, like this:

```
|   |   | | | |
-3 -2 -1 0 1 2 3
```

The real question is whether this scheme is consistent. It would be nice if numbers got bigger as you moved from left to right anywhere on the number line (again, the left side of 0 would be consistent with the right side). Is that true in the scheme above? Which is bigger, $-2$ or $-3$? Which is a bigger debt, $2$ or $3$? However, this is not the way we want to look at negative numbers. Remember, we place a high value on consistency in our number system. The consistent way is to say that if you need to add 1 to $-3$ in order to get $-2$, then $-3$ is less than $-2$ (just like 2 is less than 3 because you need to add 1 to 2 to get 3). That means $-2$ should be to the right of $-3$. Hence the number line finally came to look like this:

```
|   |   | | | |
-3 -2 -1 0 1 2 3
```

Again, what may seem perfectly obvious to us today was not nearly so obvious even a few hundred years ago.

Now that we have one number line, what we need is a second one, drawn vertical and intersecting the horizontal one at a point which is 0 on both lines. This point of intersection is called the origin. The horizontal line is called the **x-axis**, and the vertical line is the **y-axis**. Points in the plane are associated with ordered pairs of numbers, $(x,y)$. The first number, or **x-coordinate**, tells you how far to go from the origin in the horizontal direction, along the x-axis. The second number in the ordered pair, the **y-coordinate**, tells you how far to go in the vertical direction, parallel to the y-axis.

```
\begin{tabular}{c|c|c|c|c|c}
& -3 & -2 & -1 & 0 & 1 \\
\hline
-2 & & & & & \\
\hline
-3 & & & & & \\
\hline
\end{tabular}
```
One of the main goals of this procedure is to associate curves in the plane with equations. An equation containing an $x$ and a $y$ may be true if certain pairs of values replace $x$ and $y$, and false if other pairs of values are used. For instance, consider the equation $y = 2x$. The ordered pair $(3,6)$ makes the equation true, since $6 = 2(3)$; the ordered pair $(4,1)$ doesn’t make the equation true since $1$ does not equal $2(4)$. The picture of all the ordered pairs that make the equation true is called the **graph** of the equation. In this case, it can be shown that the equation $y = 2x$ has a graph which is a line.

Consider the equation $y = x^2$. The ordered pair, or point, $( -2,4)$ is on this curve; the point $(3,6)$ is not. You should be able to tell rather easily whether a given point is on a curve. What the curve looks like is a much more difficult issue. This curve is called a parabola, and is very important in mathematics. We won’t study such curves. However, before we leave them, here’s a useful observation. Two different equations may correspond to curves with the same shape. For instance, in the picture below, the two parabolas have the same shape (are similar), but because they are in different positions relative to the axes, the equations are different. At times, this is actually an advantage, one not possible in Euclid’s way of doing geometry.

\[
\begin{align*}
y &= x^2 \\
x &= y^2
\end{align*}
\]

**Slope of a Line**

Euclid’s first axiom for geometry was that two points determine a line. The fact that any two points on the line determine that line is at the basis of the most important number associated with a line, its **slope**. The slope of a line is how much the line rises (or falls) as you move 1 unit from left to right. The letter $m$ is typically used to stand for slope; some formulas for slope are these:

\[
m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y \text{ coordinates}}{\text{change in } x \text{ coordinates}} = \frac{y_2 - y_1}{x_2 - x_1},
\]

where $(x_1, y_1)$ and $(x_2, y_2)$ are any two points on the line.
For instance, if the line contains the points (2,4) and (3,6), we would compute

\[
m = \frac{6 - 4}{3 - 2} = \frac{2}{1} = 2.
\]

Notice that it does not matter in which order you use the points (as long as you are consistent, taking the points in the same order on the top and bottom of the fraction):

\[
m = \frac{4 - 6}{2 - 3} = \frac{-2}{-1} = 2.
\]

It also does not matter what two points from the line you use: the calculation of the slope of that line will always give you the same number for the slope of that line. For instance, the line containing the points (2,4) and (3,6) would also contain the point (5,10). Using (3,6) and (5,10) to compute the slope of this line, we get

\[
m = \frac{10 - 6}{5 - 3} = \frac{4}{2} = 2.
\]

The slope of a line gives very specific information about how the line is positioned relative to the x and y axes. First, it is important to agree that we shall observe lines from left to right, that is, in the direction in which x increases. With that agreement, it makes sense to talk about a line rising or falling. The connection with slope is this:

- if the line has positive slope, then it will rise from left to right;
- if the line has negative slope, then it will fall from left to right; and
- if the line has zero slope, then it is horizontal.

The more the slope differs from zero, the more steep or more nearly vertical the line will be. That is, a line with slope 2 is more steep than a line with slope 1. In the case of falling lines, a line with slope \(-2\) is more steep than a line with slope \(-1\).

Two relationships between lines are fundamental in geometry: parallel and perpendicular. How are these relationships expressed using the concept of slope? **Parallel lines have the same slope.** This should seem reasonable since parallel lines have the same direction. The slopes of perpendicular lines are harder to relate. This much seems fairly clear: if one line is rising, then a line perpendicular to it would
need to be falling, and vice versa. So for perpendicular lines, one slope would be positive and the other would be negative. The lines would also differ in their steepness. However, it turns out that there is a very precise statement we can make that incorporates these insights. The slopes of perpendicular lines, when multiplied, equal \(-1\).

For instance, a line with slope \(-2\) and a line with slope \(\frac{1}{2}\) are perpendicular.

Vertical lines are a special case. For the purposes of the homework, we will not use vertical lines. However, to be complete, we should discuss them briefly. If a line is vertical, and two points from that line are chosen, both points will have the same x coordinate, because one point will be directly above the other point. If you tried to use the formula for slope, you would find yourself trying to divide by zero. Because of this, some people prefer to say that vertical lines have no slope. While this is technically correct, I have found that some people tend to confuse "no slope" (vertical lines) with "zero slope" (horizontal lines). Also, lines which are very close to vertical have slopes like 100 or 1,000. Consequently, it seems better to me to say that vertical lines have "infinite slope." I'm not using infinity as a real number here; just as a convenient way of expressing myself.

Equations and Graphs of Lines

Lines which have slopes (lines which are not vertical) have equations that fit a particular pattern. First, we will describe that pattern; then we will study how to find the equation of a line from information given about it.

The equation of a line is an equation involving an x and a y. The points on the line correspond to ordered pairs which would make the equation true. For instance, the line discussed above which contained the points (2,4) and (3,6) has the equation \(y = 2x\). Notice that \(4 = 2(2)\) and \(6 = 2(3)\). Actually, a line can have many equations which describe it, but they are all algebraically equivalent: algebraic manipulations could be used to change from one equation to another. Other equations for the line just mentioned would be \(y - 2x = 0\) and \(2x - y = 0\).

We will discuss two general approaches or methods of finding the equation of a line. You may use either method, or any other method you may know. We will illustrate the first method before introducing the second method.

The first method uses what is often called the point-slope form of the equation of a line. If the slope of the line is \(m\) and the line contains the point \((x_1, y_1)\), then the equation of the line is

\[
y - y_1 = m (x - x_1) .
\]

If you memorize this formula, this is probably the easiest way to get an equation for a line.

Example 1 (first method): Find the equation of the line with slope 2 and containing the point (3,7).
Solution: \( m = 2 \) and \((x_1, y_1) = (3, 7)\), meaning \( x_1 = 3 \) and \( y_1 = 7 \). So the equation is

\[
y - 7 = 2 (x - 3).
\]

This is equivalent to

\[
y - 7 = 2x - 6, \text{ or } y = 2x + 1.
\]

Note that you can check your work by substituting the point \((3, 7)\) into this answer:

\[
7 = 2(3) + 1, \text{ which is true.}
\]

A second method to find the equation of a line uses what is called the slope-intercept form of the line. The word "intercept" refers to the point where the line "intercepts" or intersects the Y-axis. Since points on the Y-axis would always have an x-coordinate of 0, the Y-intercept will have the form \((0, b)\). The slope-intercept form of the line is

\[
y = mx + b.
\]

For instance, the answer to example 1 was written in this form: \( y = 2x + 1 \). This says the slope is 2 and the Y-intercept is \((0, 1)\). Now we will redo example 1 using this form.

Example 1 (second method): Find the equation of the line with slope 2 and containing the point \((3, 7)\).

Solution: \( m = 2 \) implies that the equation of the line is \( y = 2x + b \) for some as yet unknown \( b \), and \((3, 7)\) is a point on the line, so it must make the equation of the line true when substituted, i.e., when \( x = 3 \) and \( y = 7 \). Thus we have

\[
7 = 2 (3) + b, \text{ so } b = 7 - 6 = 1.
\]

So the equation of the line is \( y = 2x + 1 \).

Example 2: Find an equation of the line containing the points \((4, 1)\) and \((2, -3)\).

Solution: No matter which of the two methods you wish to use to do this problem, you must find the slope of the line first.

\[
m = \frac{\text{change in } y}{\text{change in } x} = \frac{-3 - 1}{2 - 4} = \frac{-4}{-2} = 2
\]

Method 1: \( y - y_1 = m (x - x_1) \)

\[
y - 1 = 2 (x - 4)
\]

\[
y - 1 = 2x - 8
\]

\[
y = 2x - 7
\]

Method 2: \( y = mx + b \)

\[
1 = 2(4) + b
\]

\[
1 - 8 = b \text{ or } b = -7
\]

\[
y = 2x - 7
\]
Example 3: Find an equation of a line parallel to the line \( y = 3x + 5 \) and through the point (2,4).

Solution: The equation of the line given is in slope-intercept form and therefore the 3 is in front of the x is the slope of the given line. Since the line we want is parallel, and therefore has the same slope, \( m = 3 \).

Method 1: \((x_1,y_1) = (2,4)\), so the equation of the line is

\[ y - 4 = 3(x - 2), \quad \text{or} \quad y - 4 = 3x - 6, \quad \text{or} \quad y = 3x - 2. \]

Method 2: \( y = mx + b \) becomes \( 4 = 3(2) + b \), so \( b = 4 - 6 = -2 \). So \( y = 3x - 2 \).

Example 4: Find an equation of a line perpendicular to the line \( y = 3x + 5 \) and through the point (2,4).

Solution: The slope of the given line is 3. Since the line we want is perpendicular to the given line, its slope must be the “negative reciprocal” of 3, namely \( -\frac{1}{3} \). So by method 1, \( m = -\frac{1}{3} \) and \((x_1,y_1) = (2,4)\), so the equation of the line is

\[ y - 4 = -\frac{1}{3}(x - 2), \quad \text{or} \quad y - 4 = -\frac{1}{3}x + \frac{2}{3}, \quad \text{or} \quad y = -\frac{1}{3}x + \frac{4}{3}. \]

Applications

The slope of a line represents a constant rate of change. Rates of change are often expressed in everyday language by the use of the word "per". Grocery stores and gas stations are places where we all encounter "slopes". The prices $.59 per pound for bananas or $1.69 per gallon for gasoline are examples. If \( x \) represents the amount you buy (whether bananas measured in pounds or gasoline measures in gallons), multiplying the rate ("price") by \( x \) tells you how much you will pay ("cost"). For bananas, the general formula would be \( y = .59x \). For 4 pounds of bananas, you would pay

\[ y = (.59)(4) = 2.36 = $2.36. \]

If you wanted to know how many pounds of bananas you could buy for a certain amount of money, you would substitute that amount for \( y \), and solve the resulting equation for \( x \). For instance, to find how many pounds of bananas you could buy for $2.00, you would write

\[ 2.00 = .59x, \quad \text{or} \quad \frac{2.00}{.59} = x, \quad \text{or} \quad x = 3.4 \text{ pounds of bananas.} \]

For gasoline, the general formula would be \( y = 1.69x \), if the price of gasoline is $1.69 per gallon. For 8 gallons of gasoline, the cost would be \( y = (1.69)(8) = $12.72 \). If you wanted to know how much gasoline you could purchase for $10, you would solve \( 10 = 1.69x \) for the unknown \( x \). \( x = \frac{10}{1.69} = 5.92 \) gallons.
Using such formulas demands consistency in the units used for each unknown. Since the formula usually is written without labels such as gallons or dollars, you should be careful to make sure you know what units are being used. Here’s a simple example. Suppose a florist is selling long-stemmed roses at $2.50 per rose. The formula for the cost of roses would be \( y = 2.50 \times \). How much would expect to pay for 2 dozen long-stemmed roses? (Of course, there might be a discount for buying by the dozen, but let’s ignore that for the moment.) It would be a big mistake to simply replace \( x \) by 2 and compute

\[
y = (2.50)(2) = 5.00!
\]

The quantity "2 dozen" needs to be converted to "24 roses" first, and \( x \) is replaced by 24. Now

\[
y = (2.50)(24) = 60.00.
\]

Suppose the height of a mature tree grows at a rate of 2 feet per year. Such a rate of growth is the slope of the equation which describes the growth of the tree. Suppose the tree was 9 feet tall when it first reached maturity; let \( x \) represent the number of years since then. The formula \( y = 2x + 9 \) would describe the height \( y \) of the tree \( x \) years after it reached maturity. In a graph, this would look like

![Graph of linear function](image)

Notice that it is not uncommon to have different scales on the \( x \) and \( y \) axes in practical applications. That means it is very important to label a graph carefully when you are drawing it, and very important to observe the labels carefully when you are reading a graph.

How many years past maturity would it take for this tree to reach a height of 20 feet? Replace \( y \) by 20 and solve for \( x \).

\[
20 = 2x + 9 \quad 11 = 2x \quad x = 5.5 \text{ years.}
\]

Exponential Functions

There are many situations in which the rate of change in a quantity is not a constant, but changes in proportion to some other quantity. For instance, if a certain college raises tuition at a rate of 7% per year, and tuition was $12,000 this year, then the following chart shows the growth in tuition over the next several years:
As you can see, while the increase is 7% each year, the dollar amount of the change is actually increasing each year. That’s because the 7% is applied to the current tuition each year.

Here’s a graph of the situation. In this picture, the top “line” represents the 7% per year increase situation. The “line” is really a curve. The lower line is the straight line representing what tuition would look like if it increased by a constant $840 per year.

The difference between the curve and the straight line is easier to see if a longer time period is considered. Here’s the data and the graph for a 20 year period.

If tuition were merely increased at a rate of $840 per year for 20 years, it would grow to only $28,800. Granted that’s a lot of money, but it’s a lot less than $46,436!
What kind of formula works in this case? A quick analysis of what we did in the first three lines of the chart will be sufficient to suggest the general pattern.

\[ x = 0 \quad y = 12000 \]
\[ x = 1 \quad y = 12000 + (.07) (12000) \]
\[ = 12000 (1 + .07) \]
\[ x = 2 \quad y = 12000 (1 + .07) + (.07) (12000 (1 + .07)) \]
\[ = (12000 (1 + .07)) (1 + .07) \]
\[ = 12000 (1 + .07)^2 \]

This suggests the following formula: \( y = 12000 (1 + .07)^x \). This is an example of what is called an exponential function. Notice that the unknown \( x \) is located in the exponent position. Such a function is much more difficult to handle than the linear functions we have been using. However, since there are some simple examples that are common in everyday life, and since the contrast with the constant slope of the linear functions is so significant, we will pursue our study of them.

The exponential functions we will study will all have the form

\[ y = A b^x \]

It is important to note that the \( A \) and the \( b \) represent known constants, numbers that will be given in the problem. The exponent \( x \) is applied only to the \( b \), so that \( b^x \) is calculated first, and then the answer is multiplied by \( A \).

One of the most common calculations in which exponential functions are encountered is in compound interest. If you have a savings account into which you deposit some money, and you leave that deposit as well as any interest earned in the account, an exponential function will tell you your balance at any time in the future. Here are the details. Let \( A \) = the amount of money originally deposited. Let \( r \) = the annual interest rate. For the moment, assume that the interest is marked on the account once a year. Let \( x \) = the number of years you leave the money in the account. Then \( y \) = the account balance after \( x \) years. Then the function resembles our previous example.
\[ y = A \left( 1 + r \right)^x. \]

For example, if you put $800 into an account paying 4% interest annually, and leave the money in the account, the account balance in 3 years will be

\[ y = 800 \left( 1 + .04 \right)^3 = 800 \left( 1.1249 \right) = $899.89. \]

In 20 years, the same $800 would become

\[ y = 800 \left( 1 + .04 \right)^{20} = 800 \left( 2.1911 \right) = $1,752.88. \]

Of course, another question is what inflation might have done to the buying power of that amount of money.

**Example 5:** Suppose a country’s population is growing at a rate of 4% per year. If its population was 215,000,000 in 2000, what will its population be in 2005?

**Solution:**

\[ y = A\left(1 + r\right)^x \]

In this problem, \( A = 215,000,000 \); \( r = 4\% = .04 \); and \( x = 5 \) years.

\[ y = 215,000,000 \left( 1 + .04 \right)^5 \]

\[ = 215,000,000 \left( 1.217 \right) \]

\[ = 261,655,000. \]

**Linear Programming**

Linear programming makes use of the graphs of simple linear equations to enable us to solve complicated problems dealing with the most efficient use of time, space, money and other resources. Originally developed by mathematician George Dantzig in 1947, the method became more cumbersome as the number of variables increased, especially after 20,000 variables. In 1984, Narendra Karmarkar, a mathematician born in India but working for Bell Laboratories, found a way to solve systems involving more than 500,000 variables. Big business, of course, stood in line to make use of his method to solve their distribution problems. We’ll look at a couple of simple problems; simple in the sense that we will limit the number of equations and the number variables to two. In this way, we can get a feel for the method without getting too bogged down in the algebra.

**Example #1** The Smith Tool Company makes two kinds of drill bits, Type A and Type B. (1) The company must make at least one Type B bit per hour but (2) cannot make more than 5. (3) They are unable to make more than 6 of the Type A bits in an hour’s time. They also want to make sure that (4) they do not make more Type A bits than Type B bits. (5) Type A yields a profit of $3 per bit while Type B yields a profit of $7 per bit. How many of each should they make each hour to maximize their profit?

Make sure you’ve read the problem carefully and understand it before you begin solving the problem. When you’re ready to solve the problem, begin by defining the roles of the variables. We’ll need 2 variables here as we have two types of bits.

**Solution:** 1. Define \( x \) and \( y \):

Let \( x = \) the number of Type A bits

\( y = \) the number of Type B bits
2. Write equations or inequalities that reflect the restrictions or constraints on the values of \( x \) and \( y \).

(1) \( y \geq 1 \)
(2) \( y \leq 5 \)
(3) \( x \leq 6 \)
(4) \( x \leq y \)

3. Write an expression for the profit.
(5) \( 3x + 7y \)

4. Carefully graph the inequalities in step 2.

5. Determine the coordinates of the corner points of the graph. All \( x \) and \( y \) values must be greater than or equal to 0 since it is impossible to have a negative number of drill bits.

The coordinates are: \((0, 1), (1, 1), (0, 5), \) and \((5, 5)\)

6. Substitute the coordinates of the corner points into the expression for the profit: \( 3x + 7y \)

\[
\begin{align*}
(0, 1) \text{ yields a profit of } & 3(0) + 7(1) = 0 + 7 = 7 \\
(1, 1) \text{ yields a profit of } & 3(6) + 7(1) = 3 + 7 = 10 \\
(0, 5) \text{ yields a profit of } & 3(6) + 7(5) = 0 + 35 = 35 \\
(5, 5) \text{ yields a profit of } & 3(5) + 7(5) = 15 + 35 = 50
\end{align*}
\]

Since the point \((5, 5)\) yields the highest profit the company should make 5 of the Type A bits and 5 of the Type B.

Of course, having done this and looking back at the problem, the solution is obvious. But that is not always
Example #2  As an officer manager, you want to buy some filing cabinets to store some hard copies of your computer files. Your budget is somewhat limited, $1400, and you are looking at two cabinets, Type A which costs $100 and Type B which costs $200. You have measured the floor space available and found that you have 72 square feet available for the cabinets. The catalog descriptions indicate that Type A requires 6 square feet and Type B requires 8 square feet. Type A cabinet will provide 8 cubic feet of storage space and Type B will provide 12 cubic feet. How many of each type cabinet should you purchase to maximize your storage space?

Solution  Let $x = \text{number of Type A cabinet purchased}$

$y = \text{number of Type B cabinet purchased}$

Restrictions:  

$100x + 200y \leq 1400$  or  $x + 2y \leq 14$

$6x + 8y \leq 72$  or  $3x + 4y \leq 36$

Storage Space:  $8x + 12y$

Graph:

Corner Points:  $(0, 7), (12, 0), (8, 3)$

Storage Space:  

$(0, 7)$ provides 84 cubic feet

$(12, 0)$ provides 96 cubic feet

$(8, 3)$ provides $64 + 36 = 100$ cubic feet.

Solution:  You should purchase 8 of Type A and 3 of Type B at a cost of $1400 in order to maximize your storage space.

Something to think about:  As our problems increase in complexity, the number of variables increase which, of course, poses a problem when we graph our inequalities. For example, if we have three variables then we need three axes which means we are graphing in 3-dimensional space. That is difficult to do on a flat sheet of paper. Four variables implies 4 axes. Can you visualize 4 dimensions? 20,000? 500,000? Dr.Karmarker’s method enables big businesses to solve systems of equations involving thousands of variables, allowing these businesses to be as efficient as possible.
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Note: Use graph paper for your sketches. You can download then print graph paper at http://incompetech.com/graphpaper/lite/ or do a Google search for “graph paper”

1. Sketch a graph for each line and find the slope of the line determined by the points:
   a. (1,2) and (3,6)       b. (2,1) and (4,0)       c. (–1, – 2) and (– 3,– 4)   d. (–1,3) and (2,0)

2. Find the equation of each line, then sketch the graph for sub-problems a, c, e, and g.
   a. with slope 3 and through the origin.
   b. with slope 2 and through (0,3)
   c. with slope – 2 and through (0,2)
   d. determined by (2,1) and (4,3)
   e. determined by (–1,1) and (2,– 3)
   f. parallel to \( y = – x + 3 \) and through the origin
   g. parallel to \( y = 2x – 3 \) and through (1,1)
   h. parallel to \( y = – 2x + 4 \) and through (–2,– 3)

3. Find the coordinates of the point of intersection of the line and the Y-axis.
   a. \( y = 3x + 5 \)       b. \( y = 2x – 3 \)       c. \( x + 2y = 6 \)

4. Find the coordinates of the point of intersection of the line and the X-axis.
   a. \( x + 2y = 6 \)       b. \( y = 2x – 4 \)       c. \( y = x + 3 \)

5. Find the equation of the line
   a. parallel to \( y = 2x – 3 \) and through (4,1)
   b. perpendicular to \( y = x – 3 \) and through (2,4)
   c. perpendicular to \( y = \frac{1}{2} x + 1 \) and through (3,0)
   d. perpendicular to \( y = 3x \) and through (– 2,– 1).

6. Find the equation of the line
   a. parallel to \( y = 3x + 1 \) and through (2,–1)
   b. perpendicular to \( y = 2x – 3 \) and through (–1,4)
   c. perpendicular to \( y = – x + 1 \) and through (2,3)
d. perpendicular to $y = \frac{1}{3} x$ and through $(-2, -5)$.

7. Tomatoes are $1.39$ per pound. How much will 5 pounds of tomatoes cost?

8. Drinking water costs $1.19$ per gallon. How much will 3 gallons of drinking water cost?

9. Hamburger costs $1.89$ per pound. How much will a 10 ounce package cost?

10. Imitation crab salad costs $1.99$ per pound. How much will a 16 ounce package cost?

11. A city's population is growing at a rate of 17,000 people per year. In 2000, its population was 185,000. What will its population be in the year 2010?

12. A church's membership is growing at a rate of 100 members per year. In 2007, it's membership was 1350. What will its membership be in 2015?

13. Bartlett pears cost $0.89$ per pound. How many pounds of Bartlett pears can you buy for $2.50$?

14. Premium gasoline costs $3.39$ per gallon. How many gallons of premium gasoline can you buy for $25.00$?

15. A certain college plans on increasing its tuition at a rate of 5% annually. Its tuition is currently $13,700. How much will its tuition be in 6 years?

16. In a certain city, the cost of housing is increasing at a rate of 9% annually. This year the typical house cost $454,000. How much will the typical house cost in 4 years?

17. Suppose you put $700 into an account paying 3% interest annually, and leave the money in the account for 5 years. Find the account balance.

18. Suppose you put $1200 into an account paying 6% interest annually, and leave the money in the account for 10 years. Find the account balance.

19. A country's population is growing at an annual rate of 3%. In 2005, its population was 54,000,000. What will its population be in the year 2015?

20. The number of people who are members of a Christian church in a certain country is growing at an annual rate of 18%. In 2008, the number of members of a Christian church was 3,600,000. What will the number be in 2015 (7 years)?
EXTRA CREDIT: Problems 21-24. Use Graph Paper

21. Graph each of the following inequalities in the same coordinate plane and determine the coordinates of the corner points.
   a. \( x \geq 0 \)
      \( y \geq 0 \)
      \( 2x + 5y \leq 20 \)
      \( 2x + y \leq 12 \)
   b. \( x \geq 0 \)
      \( y \geq 0 \)
      \( 3x + 2y \leq 12 \)
      \( x + 2y \leq 8 \)

22. Maximize the expression \( 30x + 20y \) subject to the constraints in problem #21a.

23. An airline agrees to provide space, on a special occasion, for at least 160 first-class passengers and at least 300 tourist-class passengers. It must use at least two of its Type A planes. Each type A plane has room for 20 first-class seats and 30 tourist seats. Each type B plane has room for 20 first-class seats and 60 tourist seats. The flight will cost the airline \$1000 for each type A plane and \$1500 for each type B. If the cost is to be held to a minimum, how many of each kind of plane should the company use?

24. George, who is ill, must take vitamin pills daily. Each day he must have at least 16 units of vitamin A, 5 units of vitamin B1 and 20 units of vitamin C. He can choose between pill #1 which costs 20 cents each and provides 8 units of vitamin A, 1 unit of vitamin B1 and 2 units of vitamin C and pill #2 which costs 30 cents each and provides 2 units of vitamin A, 1 unit of vitamin B1 and 7 units of vitamin C. How many of each pill should he take on a daily basis to satisfy his needs and minimize his costs?
Selected Answers:

1. a. 2  
   b. \(-\frac{1}{2}\)

2. a. \(y = 3x\)  
   b. \(y = 2x + 3\)  
   d. \(y = x - 1\)  
   f. \(y = -x\)  
   g. \(y = 2x - 1\)

3. a. (0,5)  
   b. (0,-3)  
   c. (0,3)

4. a. (6,0)  
   b. (2,0)  
   c. (-3,0)

5. a. \(y = 2x - 7\)  
   b. \(y = -x + 6\)  
   c. \(y = -2x + 6\)  
   d. \(y = -\frac{1}{3}x - \frac{5}{3}\)

6. a. \(y = 3x - 7\)  
   b. \(y = -\frac{1}{2}x + \frac{7}{2}\)  
   c. \(y = x + 1\)

7. $6.95

9. $1.18

11. 355,000 people

13. 2.8 pounds

15. $18,359

17. $811.49

19. 72,571,485 people

20. 11,467,706 people

21. a. (0,0), (0,4), (5,2), (6,0)  
   b. (0,0), (0,4), (2,3), (4,0)

22. 190 at (5,2)