CHAPTER 25: Set Theory and Infinite Sets

> ...there was once in man true happiness of which there now remain to him only the mark and empty trace, which he in vain tries to fill from his surroundings,.... But these are all inadequate, because the infinite abyss can only be filled by an infinite and immutable object, that is to say, only by God Himself.[128]

The following discussion of set theory and infinite sets is foundational to much of modern mathematics, and at the same time is in part quite esoteric and even controversial. As another indication of the relationship of mathematics to other disciplines, I will follow the development of set theory in Scaling the Secular City by Professor J.P. Moreland.[129] In this book, Dr. Moreland's discussion of set theory and infinity is essential background for his apologetics discussion of a proof for the existence of God. I refer the interested reader to his book for the details of the apologetics discussion.

Sets

A **set** is a collection of objects called its **members**. In most of our examples, the objects will be numbers. Sets will often be designated by capital letters. Sets can be specified or defined in two ways: 1) by means of a list of the members in the set; or 2) by means of a rule by which you can determine whether or not an object is a member of the set . The following examples illustrate these two possibilities.

Example 1: 1) {2,4,6,8,10}
2) the set of even numbers less than 12
3) the set of even numbers

The set A is a **proper subset** of a set B if every member of A is a member of B, but B contains at least one member not in A.

Example 2: 1) {2,4} is a proper subset of {2,4,6,8,10}
2) {2,4,6,8,10} is a proper subset of the set of even numbers
3) the set of positive even numbers is a proper subset of the set of whole numbers

Two sets A and B are **equal**, written A = B, if and only if A and B have exactly the same members.

Example 3: 1) {2,4,6,8,10} = the set of even numbers less than 12
2) {1,3,5} = {3,1,5}

Two sets A and B are have the **same cardinality** (or, "have the same number of elements"), written $A \approx B$ if and only if there is a 1–1 correspondence between A and B. That is, if and only if the members of A and B can be paired in such a way that each member of A is paired with one and only one member of B, and vice-versa.

Example 4: 1) $\{1,2,3,4,5\} \approx \{2,4,6,8,10\}$ because 1 <-->2, 2<-->4, 3<-->6, 4<-->8, 5<-->10 is a one-to-one correspondence between the two sets (there are other one-to-one correspondences).
2) {2,3,4,5} is not equivalent to {2,4,6,8,10} because there is no one-to-one correspondence between these two sets.

The set A is **finite** if it can be put into one-to-one correspondence with {1,2,3,....,n} for some whole number n. Now recall that one of Euclid's general axioms was: The whole is greater than any of its parts. Here's the set theory version of that statement: A finite set cannot be put into one-to-one correspondence

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[128]cited in Eli Maor, To Infinity and Beyond, Birkhauser, Boston 1987, p.131.
[129]J. P. Moreland, Scaling the Secular City, pp. 19-32.

with any of its proper subsets. So we see that when Euclid thought about things made up of parts he assumed as axiomatic that the number of parts was finite.

Example 5: Consider the set A = {2, 4, 6, 8, 10}. I claim that since A can be put into one-to-one correspondence with {1,2,3,4,5}, A is finite. And since A is finite, I can pick any proper subset of A, for instance {2, 4, 6, 8}, and there is no one-to-one correspondence between A and {2,4,6,8}.

Potential and actual infinity

Sets which are not finite are sometimes described as **potentially infinite**. This means that the list of members of the set always could be increased from its current size. At any moment in time, the list would be finite. But any such finite list of members of the set fails to include some other members of the set.

The set of counting numbers 1,2,3,... could be thought of as potentially infinite: you can always add more numbers to the list if and when you need them. At the moment, the list goes to 3. If you need to talk about the national debt of the United States, you can extend the list into the trillions. One of the advantages of a place value notation for numbers (and exponent notation) is that it makes extending the list of whole numbers very easy. In a system like the early Egyptians used, larger numbers required new symbols.

Here's another mathematical example of a set which some people would call potentially infinite. The list of known prime numbers 2, 3, 5, 7, 11, 13, ... is finite; that is, if you checked with all the people alive today, there is a largest prime number that any of them could tell you. However it has been known since the time of Euclid that there is no largest prime number (See Chapter 4). The only problem is that finding the next prime number isn't nearly as easy as writing the next counting number.

Now recall the lines of Euclidean geometry. Euclid did not assert that lines were actually infinitely long. Rather, he said that they could always be extended, but at any given time were finite in length. The lines do not have a fixed finite length, because you could always make them longer than any fixed number.

Now potentially infinite sets are both useful (like having the potential to get all the numbers you will ever need or being able to extend a line) and also somewhat problematic.

In reality, when we gave answers to problems like this in the previous chapter, we were using mathematical concepts and procedures that avoided doing “an infinite amount of adding.” Calculus includes a careful and consistent theoretical treatment of infinite series along with many practical applications of them. The details are unimportant for our purposes at the moment. The issues of infinity relating to infinite series were discussed by mathematicians, scientists, and philosophers for about 150 years before being put on a firm logical basis solely within the realm of potential infinity. We may talk informally about “adding an infinite number of numbers”, but mathematicians in reality believe in no such thing.

In contrast to potential infinity which is at all times finite, **actual infinity** is thought of as having “actually arrived” at infinity. The set is not growing larger and larger, but has already reached “infinity”. There is no longer "potential"; the set is complete. For instance, the set of counting numbers in this case is thought of as a completed list {1,2,3,4,...} which actually contains an infinite number of numbers. We need some definitions and further discussion to clarify this meaning. In the discussion below, “infinite” will be used in the sense of “actually infinite”.

A set is **infinite** if it cannot be put into one-to-one correspondence with the set {1,2,...,n} for any n. A problem with this definition is that it has a negative approach in it. It suggests that you would need to try something an infinite number of times and it would never work. A more direct approach to infinite sets would be nice, and our previous discussion of finite sets provides just what we need.

An equivalent definition for an infinite set is this: a set is **infinite** if it can be put into one-to-one correspondence with a proper subset of itself. This is the opposite of a property we noticed was true for finite sets, so it makes sense that it would tell us exactly what infinite ("not finite") sets are like. Let's consider an example.

Example 6: {1, 2, 3, 4, 5, ... } = E

{2, 4, 6, 8, 10 . . .. } = F

There is a one-to-one correspondence between the two sets: match a number in E with the number in F directly below it. In general, the number n in E would be matched with 2n in F.

This means E and F have the same cardinality, i.e., E and F have the same number of elements. We write this relationship $E \approx F$. We need a new symbol for this number; mathematicians have used the first letter of the Hebrew alphabet, aleph, with a zero subscript, $\aleph_0$, read "aleph null", to represent this "quantity".

Now if you are thinking that E and F can't have the same number of members since "E contains all the members of F and infinitely more, so E must be bigger than F", about the only response I can give is that you are not being consistent. All we did was extend to infinite sets a definition of equivalent sets that worked quite well (fit our intuition) for finite sets. The basic concept is the same, though it's implications in the realm of the infinite are sometimes a bit different. But would it be reasonable to expect anything else? Isn't it "reasonable" that the infinite would be somewhat different that the finite?

Example 7: This is a non-mathematics example with which some of you may be familiar. Do you know the old church hymn, "Amazing Grace"? The last verse goes like this:

When we've been there ten thousand years
Bright shining as the sun,
We've no less days to sing God's praise
Than when we'd first begun.

John Newton, the author of this hymn (not to be confused with Isaac Newton the mathematician and scientist) is suggesting that the number of days we will spend in heaven from the time we enter is the same as the number of days we will spend in heaven after we have already been there 10,000 years! In our notation, this says $\aleph_0 - 10{,}000 = \aleph_0$.

Notice that Euclid's axiom "the whole is greater than any of its parts" doesn't hold for infinite sets; for infinite sets, the whole is <u>not</u> greater than each of its proper parts. In fact, the essence of being infinite is that "the whole is greater than each of its proper parts" is not true. Instead, in the infinite realm, the whole is equivalent to some of its proper parts.

Does this mean that infinite sets are "absurd" or "irrational"? No, of course not; they are just different, like fractions are different from whole numbers, or irrationals from rationals, or imaginary numbers from real ones.

Operations with sets

Since the concept of "set" is more fundamental, at least in modern mathematics, than is the concept of "number", we are going to define operations with numbers, like addition, in terms of operations with sets. So we need to talk about ways to put two sets "together".

The first operation is the "union" of two sets. "Union" here has much the same idea as it does in American politics: the Union is formed by putting a group of states into one big country. The **union** of A and B is the set whose members are either members of A or members of B. It is denoted by $A \cup B$.

Example 8: $A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$ $A \cup B = \{1, 2, 3, 4, 5, 7\}$
Note that even though 3 is in both A and B, it is only listed once in $A \cup B$.

The second operation is the "intersection" of two sets. "Intersection" means much the same as the intersection of two roads: it's where they cross or overlap. The **intersection** of A and B is the set whose members are members of both A and B. It is denoted by $A \cap B$.

Example 9 $A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$ $A \cap B = \{3\}$

If A and B have none of the same members, they are called "disjoint". This means their intersection contains no members. This brings us to a concept analogous to the number zero. It is common today to use the notation " $\varnothing$ "; this is called the "empty set".

Example 10: $A = \{1,2\}$ $B = \{5,7,8\}$ $A \cap B = \varnothing$

The operations of union and intersection of sets have some very important properties. For instance, one property they share with addition and multiplication is that order doesn't matter, i.e., $A \cup B = B \cup A$ and $A \cap B = B \cap A$. We won't need to pursue such properties any further for our purposes.

Addition of whole numbers

So you believe 2+2=4. Why? If you had to explain why in terms of more fundamental concepts, how would you do it? How does addition really work? Here's a simple approach through the use of sets.

To add two numbers "a" and "b", we start by finding two disjoint sets A and B containing a and b members. Then, quite naturally, a + b = number of members of $A \cup B$. If you owned 12 sheep, and your neighbor owned 16 sheep, and you put them in one field and counted sheep, you would find that 12 + 16 = 28. Here's the more abstract approach:

Example 11: $2 + 3 = ?$

$A = \{x,y\}$ $B = \{r,s,t\}$

$2 + 3$ = the number of elements in $A \cup B$.

= the number of elements in $\{x,y,r,s,t\}$

= 5.

All of the properties of addition can be derived from this definition of what addition is. For instance, the fact that a + b = b + a is proved by using the fact that $A \cup B = B \cup A$.

Now all of this should make good sense for regular counting numbers, i.e., finite numbers. It seems reasonable, then, that it at least makes sense to try to use this definition of addition of numbers with infinite numbers. There's no good reason for simply asserting that addition with infinity makes no sense. How would you know that? So, let's give it a try.

Let's start with something simple. How about $1 + \aleph_0$? We proceed as above: we look for two disjoint sets, the first having one member and the second having $\aleph_0$ members. Let $A = \{1\}$ and $B = \{2, 3, 4, 5, \ldots\}$. Then $1 + \aleph_0$ = the number of elements in $A \cup B$ = the number of elements in $\{1, 2, 3, \ldots\} = \aleph_0$.

So $1 + \aleph_0 = \aleph_0$. This is different, but not illogical. Just different. Would you have expected anything less? Infinity is different, so it should not be surprising if the rules about adding would look different. But "different" isn't "nonsense". $\sqrt{2}$ wasn't a rational number, but that didn't make it illogical. Gauss's non-Euclidean geometry was different, but it wasn't nonsense.

How about a harder problem? What is $\aleph_0 + \aleph_0 = ?$ Again, we search for two disjoint sets. Let's use $A = \{1, 3, 5, 7, \ldots\}$ and $B = \{2, 4, 6, 8, \ldots\}$. Then

$\aleph_0 + \aleph_0$ = the number of members in $\{1, 2, 3, 4, \ldots\} = \aleph_0$. Yes, $\aleph_0 + \aleph_0 = \aleph_0$.

By the way, it follows logically that $\aleph_0 + \aleph_0 + \aleph_0 = \aleph_0$. Three infinities, added together, give you the same infinity. Now, where have you ever heard of such a thing before? Yes, it appears that how adding infinities works in mathematics is somewhat analogous to the Trinity. Notice I said, "somewhat analogous". God is not the same as infinity in mathematics. On the other hand, the doctrine of the Trinity is sometimes criticized as being nonsense. Well, it may seem somewhat mysterious to our finite minds, but that doesn't make it nonsense.

The mathematician who first developed these ideas about infinity was Georg Cantor, who lived about 100 years ago. Cantor was a mathematician whose researches into standard problems in mathematical physics led him to the concept of infinite sets which he regarded as actually infinite. Cantor's ideas were criticized and rejected by many mathematicians, and he was persecuted for his views. For his part, Cantor believed that his study of infinite numbers was a religious quest. Here are some of his words. (He used the word "transfinite" to describe numbers that are infinite.)

> "Nature makes frequent use of [the actual infinite] everywhere, in order to show more effectively the perfections of its Author."[130]
>
> "I entertain no doubts as to the truth of the transfinites, which I have recognized with God's help. . . "[131]
>
> "The absolute [God] can only be acknowledged and admitted, never known, not even approximately."[132]

Cantor believed that infinite numbers were a necessary part of mathematics. As you can see, he firmly believed that God had aided him in the study of infinite numbers. He asserted that a whole theory of infinite numbers could be developed which made sense and was understandable. But he was also very clear to separate the infinity of mathematics from the Infinite of theology. God is the Almighty Creator; the actual infinity of mathematics is merely one of His creations.

Here are some assertions that I believe are helpful to make at this point to clarify the relation ship between mathematics and theology.

1. God is "infinite".
   The typical approach used in explaining this statement is to say what God is not: He is not limited by time and space, He is not limited in power, etc. That is, He is not finite like us. The problem with this is that it really starts with us, with the finite. As we saw above in the mathematical context, we can start with finite sets and define infinite sets as the ones that are not finite. OR, we could start with infinite sets. In some ways, that really is preferable in mathematics. In theology, it definitely is very important to begin with God.

2. We are finite, fallen creatures made in the image of the infinite Creator.
   We are made in the image of God, meaning He made us in some ways like Himself. On the other hand, we are limited by our finiteness, the fact that we are created beings. We are also, at least at the moment, limited by our fallenness. Sin has had far-reaching consequences. One day these will no longer affect us, but for now some of these consequences have only begun to be removed.

3. Christian theology is rational, but God is suprarational.
   God's thinking is "higher" than our thinking, although ours is "in His image", and therefore not totally different than His. Theology is our human attempt to state God's truth in our terms.

4. We can know sufficient truth about God, but not the whole truth.

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[130]Quoted in: Dauben, Joseph. Georg Cantor: His Mathematics and Philosophy of the Infinite. Princeton University Press, Princeton, 1979, p. 124.
[131]Ibid. p. 147.
[132]Quoted in Moore, A. W. The Infinite. Routledge, London, 1991, p. 128.

> The whole truth about God is of necessity beyond our comprehension. But that doesn't mean we can't know anything true about God. In fact, He is clear in telling us that the Bible is adequate to accurately communicate to us what we need to know about Him.

Back to the mathematics of infinity. You might ask, "OK, I follow you so far. Are we done?" Not quite. Recall that in Flatland, after A. Square was convinced about the existence of a third dimension, he reasoned that there should be a fourth dimension, a fifth dimension, ad infinitum?

Think back to finite sets for a moment. Earlier we talked about a set being a proper subset of another set.

Theorem 1: The set of all proper subsets of a finite set is larger than the set itself.

Example 12: If B = { 2, 4, 6 }, then the set of all proper subsets of B would be { {2}, {4}, {6}, {2,4}, {2,6}, {4,6}, {Ø} }. This set contains 7 members, whereas the original set B contained only 3.

Hopefully, one example is enough to convince you that this would always happen for finite sets.

Since it may be helpful to understand the proof below, let me elaborate on the example in a way that may seem unnecessarily complicated at the moment. If I attempted to establish a 1-1 correspondence between the set { 2,4,6 } and its set of proper subsets, I might write:

2 <--> {4}   4 <--> {2,4}   6 <--> {6} .

The observation we need to make is that some numbers, like 2, are matched to sets that don't include them, whereas other numbers, like 4 and 6, are matched to sets that do include them.

Since the theorem above is true for finite sets, it shouldn't seem unreasonable or irrational to think the corresponding result might be true for infinite numbers. Consider an example. If we start with the set {1,2,3,4,5,....}, which has $\aleph_0$ members, and we form the set of all proper subsets of this set, the new set we just formed would have a certain (infinite) number of members. What number would that be? Cantor's answer is that that number is not $\aleph_0$; it's larger than $\aleph_0$. What's larger than infinity, you say? Well, the point is that "infinity" is not synonymous with "biggest possible number".

Theorem 2: The set of all proper subsets of any set has a larger cardinal number than the original set.

Proof: Let A be the original set. Let S be the set of all the proper subsets of A. We want to prove that there is no 1-1 correspondence between A and S, which implies that the cardinal number of S must be larger. (There are actually some technical difficulties which we are omitting, only because they would be too time consuming.) We assume to the contrary that there is a 1-1 correspondence between A and S.

Each element a in the set A is matched with some set Y in S. Form a set W of all the elements a from A which are matched to a set S which does not contain a. Since W is a subset of A, some element of A is matched to W. Call this element of A by the designation z.

Now z is not an element of W because W consists precisely of the elements of A which are not in the set to which they are matched. So z must be an element of W. But if z is an element of W, it must not be in the set to which it is matched. Since it is matched to W, z must not be in W. We have a contradiction.

Thus our original assumption must be incorrect. That is, there cannot be a 1-1 correspondence between A and S. This concludes the proof.

This theorem leads to the following conclusion, which may seem surprising to some, but perhaps not to others.

Theorem 3: There is an unending progression of larger and larger infinities! There is no largest number.

And with Cantor, we affirm that God is beyond them all.

Oh the depth of the riches of the wisdom
And knowledge of God!
How unsearchable his judgments,
And his paths beyond tracing out!

"Who has known the mind of the Lord?
Or who has been his counselor?"
"Who has ever given to God
That God should repay him?"

For from him and through him and to him are all things.
To him be the glory forever! Amen.[133]

Application of Sets

A survey of 80 Biola freshman revealed the following results:

36 were enrolled in a math class (Math 120?)

32 were enrolled in an English class

30 were enrolled in a history class

16 were enrolled in an English class and a history class

14 were enrolled in math and English

16 were enrolled history and math

6 were enrolled in all three

By simply looking at the above results are you able to answer questions such as the following:

Were there students in the survey who didn't take any of the three classes? If so, how many?

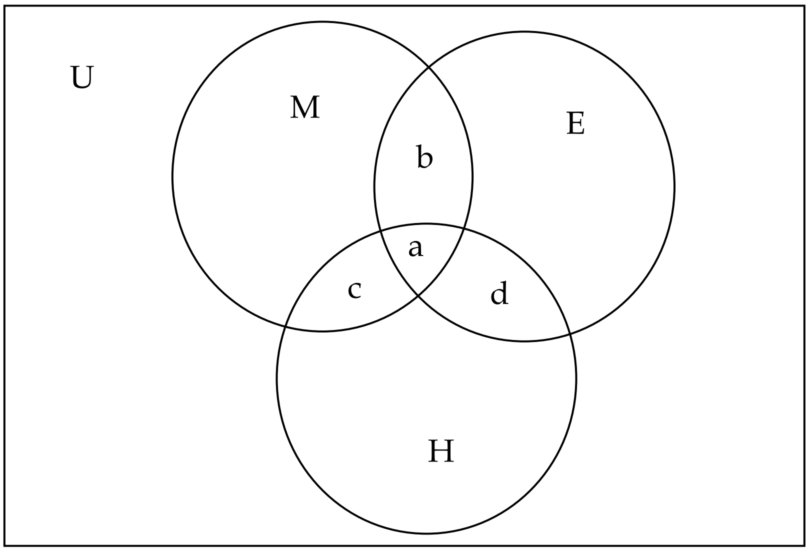
How many students took only math?

How many students took math and history but did not take English?

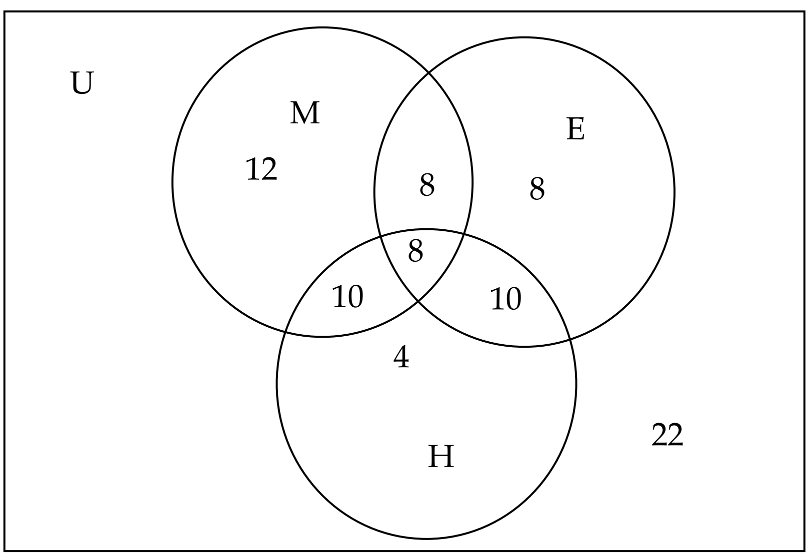
Doubtful, but making use of sets and the set operations of union and intersection will allow us to easily answer such questions.

[133] Romans 11:33-36

Our Universe Set, U, is made up of all the students surveyed, Set M contains all those taking math, Set E those taking English and Set H those taking history. The provided information indicates that there is an intersection of each of these sets as reflected in the diagram below, called a Venn diagram.



Area *a* represents those students taking all three classes, areas *a* and *b* together represent those taking math and English which means that since $a + b = 14$ and $a = 6$ then $b = 8$. With this understanding, we can fill in each of the areas with the appropriate number of students as illustrated below.



We can now answer the questions without too much difficulty.

1. How many of the students took none of the three classes?

   Solution: $80 - (a + b + c + d + e + f + g) = 80 - 58 = 22$

2. How many took only mathematics?

   Solution: $36 - (a + b + c) = 36 - 24 = 12$

3. How many took math and history but did not take English?

   Solution: $a + c$ = those who took math and history, but those in area *a* took English as well, therefore the answer is $c = 8$

CHAPTER 25: Sets and Infinite Numbers **Homework**

1. Which of the following are proper subsets of {1,2,3,4,5,6}?

   a. {1} b. {7} c. {2,4,6} d. {3,9}

2. Write all the proper subsets of

   a) {1,2} b) {a,b,c} c) {2,4,6,8}

3. Which of the following sets are infinite?

   a. {1,4,9,16,....}

   b. $\{1, 10, 100, \ldots, 10^{10000}\}$

   c. the set of all grains of sand on all the beaches of the earth.

   d. the set of all ordered pairs of numbers that make $y = 3x + 5$ a true statement

4. Display a 1 – 1 correspondence between {1,2,3,4} and {a,b,c,d}.

5. Display a 1 – 1 correspondence between {1,2,3,4,....} and {1,3,5,7,....}.

6. {1,2,....,100} is a proper subset of {1,2,3,....}. Can you find a 1 – 1 correspondence between the two sets? Does this contradict the statement that {1,2,3,....} is infinite?

7. How many members does each set have?

   a. {1,2,3,....,100}

   b. {1,2,3,....}

   c. {3,4,5,....}

   d. { 5,10, 15,20....}

8. For A = {2,4,5,8,9} and B = {3,5,7,9}, find

   a. $A \cup B$ b. $A \cap B$

9. For A = {2,4,5,6} and B = {1,3,7}, find

   a. $A \cup B$ b. $A \cap B$

10. Use sets to compute

    a. $3 + 1$ b. $2 + 4$

11. Use sets to compute

    a. $2 + \aleph_0$ b. $3 + \aleph_0$

12. Answer the following.

    a. $7 + \aleph_0 =$

    b. $\aleph_0 + \aleph_0 + \aleph_0 + \aleph_0 =$

    c. $2 \bullet \aleph_0 =$

    d. $\aleph_0 - 1 =$

    e. $\aleph_0 - 10000 =$

EXTRA CREDIT: #13, #14

13. For each of the following, draw a Venn Diagram to understand the problem and answer the accompanying questions.

    a. Interviews of 150 people concerning their Sunday morning habits yielded the following: 80 attended church, 40 attended Sunday School and 25 attended both.

        1) How many attended church only?
        2) How many attended neither?

    b. In a survey of 500 Biola students, it was found that 200 students were enrolled in Math 120, 180 were enrolled in Bible, 170 in Science, 40 in Math 120 and Bible, 45 in Bible and Science, 50 in Math 120 and Science and 14 in all three.

        1) How many students were enrolled in none of these classes?
        2) How many were taking only Math 120?
        3) How many were taking at least two of the classes?
        4) How many were taking Math 120 and Bible but were not taking Science?

    c. A political party is in the process of selecting a candidate for a statewide office. Three candidates, A, B and C are under consideration. A survey of 300 party members is conducted to determine which of the three candidates they could vote for with the following results: 130 selected A, 110 selected B, 160 selected C, 80 selected A and B, 65 selected A and C and 40 selected B and C. 27 selected all three.

1) How many selected only A?
2) How many did not select a candidate?
3) How many selected B and C but did not select A?

14. A survey of 500 farmers showed that 125 farmers grew only wheat, 110 farmers grew only corn and 90 grew only oats. 200 farmers grew wheat, 60 grew wheat and corn, 50 grew wheat and oats, 180 grew corn.

1) How many grew at least one of the three?
2) How many grew all three?
3) How many did not grow any of the three?
4) How many grew exactly two of the three?

Answers:

1. a, c
2. a. {1}. {2}, Ø   b. {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, Ø
   c. (Hint: there are 15 of them)
3. a. infinite   b. finite
4. One correct answer is: 1<-->a, 2<-->b, 3<-->c, 4<-->d
6. No, there is no such 1 – 1 correspondence. Saying that {1,2,3,...} is infinite says that it can be put into 1 – 1 correspondence with at least one of its proper subsets, not with all of them.
7. a. 100   b. $\aleph_0$
8. a. {2,3,4,5,7,8,9}   b. {5,9}
10. a. A = {a,b,c}   B = {d}
    3 + 1 = number of members of $A \cup B$ = {a,b,c,d}, namely 4
   b. A = {a,b}   B = {u,v,w,x}
    2 + 4 = number of members of $A \cup B$ = {a,b,u,v,w,x}, namely 6
11. a. A = {1,2}   B = {3,4,5,...}
    $2 + \aleph_0$ = number of members of {1,2,3,4,....}, namely $\aleph_0$
   b. A = {1,2,3}   B = {4,5,6,7,.....}
    $3 + \aleph_0$ = number of members of {1,2,3,4,....}, namely $\aleph_0$
12. a. $\aleph_0$ b. $\aleph_0$ c. $\aleph_0$, since $2 \bullet \aleph_0 = \aleph_0 + \aleph_0$
   d. $\aleph_0$ since $\aleph_0 = 1 + \aleph_0$, and we can subtract 1 from both sides.
   e. $\aleph_0$